elevate

# YEAR 11 MATHS ADVANCED

## MODULE 3

## LESSON ONE

# THEORY
# Motion

Last lesson, we learnt about Trigonometric Identities and Equations, including complementary angle identities and the Pythagorean Identity, using them to solve and manipulate a number of problems involving trigonometric identities.

Today, we'll be moving to a completely different topic: rates of change. This topic revolves around analysing and calculating the motion of certain objects, such as a travelling car or a flying baseball. This includes exploring graphs of functions and looking at an object's speed, velocity, or acceleration in distance-time graphs.

In today's lesson we will:

- Estimate rates of change;
- Distinguish between calculating average and instantaneous speed; and
- Analyse linear and non-linear functions modelling motion.

## Estimating Change

**4.1.1 Define the average rate of change of $y$ with respect to $x$ for a function $y = f(x)$ over the domain $[a, b]$ as $\frac{\Delta y}{\Delta x} = \frac{change\ in\ y}{change\ in\ x}$, that is $\frac{\Delta y}{\Delta x} = \frac{f(b)-f(a)}{b-a}$, and recognise $\frac{f(b)-f(a)}{b-a}$ as the gradient of the secant through $(a, f(a))$ and $(b, f(b))$ on the graph of $y = f(x)$**

The average rate of change describes how one quantity changes in relation to another. In graphs that describe two quantities, this can be written mathematically as:

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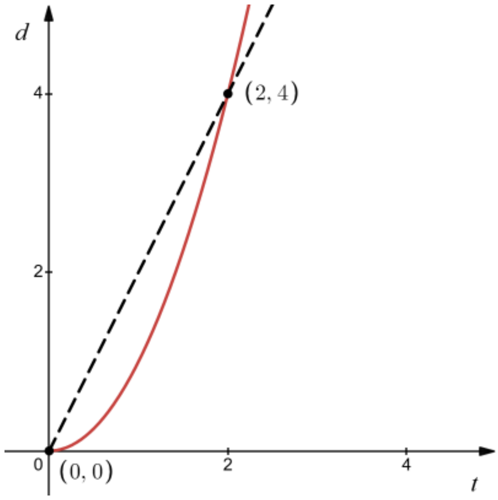
In most cases, we will encounter an object travelling a distance over a certain period of time.

The object's average rate of change would describe its speed, a measure of its distance over a given time interval.

If we were to graph this on a function with time as our x-axis and distance as our y-axis, the gradient of the secant between the endpoints of an interval would be equal in value to the rate of change over the interval.

# THEORY




For example, to find the rate of change over the interval $t = 0$ to $t = 4$, we can find the gradient of the secant drawn between the two points.

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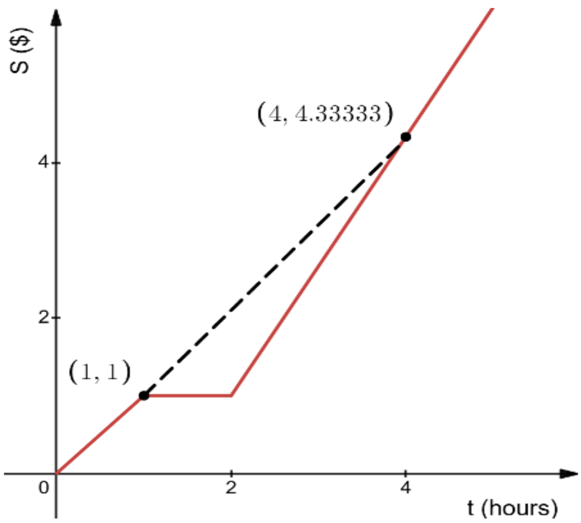
*PRACTICE QUESTIONS*

**Find the average rate of change of the following functions.**

**(1 mark)**

a.

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# THEORY

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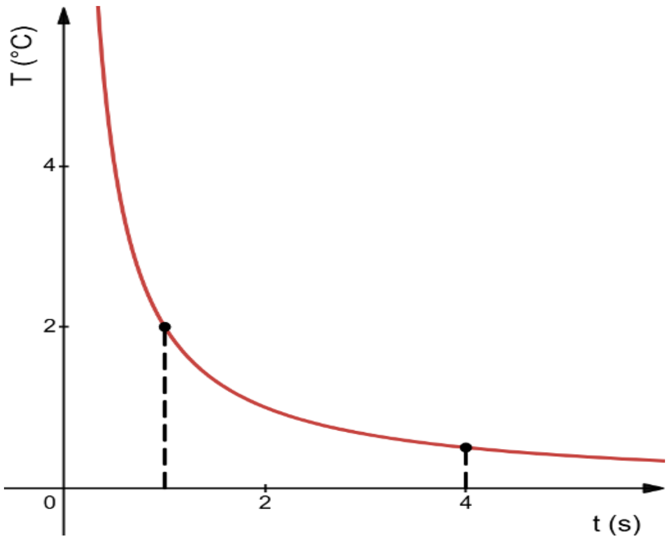
**(2 marks)**




b.

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## Motion

### 4.1.2 Recognise speed as a rate of change of distance with respect to time

As per the definition of motion being the process of moving or changing position, rates of change are heavily involved, meaning we can apply what we've learnt so far to any scenario with a moving object!

If you've dabbled in physics before, you may have heard of terms such as displacement and velocity. These are very similar concepts to distance and speed, but what distinguishes them is direction.

Across the Year 11 Advanced syllabus, we'll be faced with questions involving displacement and velocity, so it's important to differentiate between these terms!

# THEORY

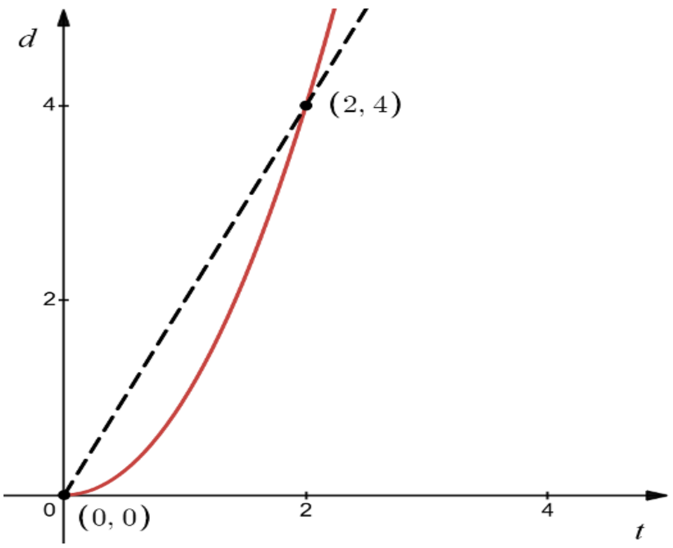
| Term | Definition |
|---|---|
| Distance | |
| Displacement | |
| Speed | |
| Velocity | |
| Acceleration | |

## Average Speed

**4.1.3 Use the definition for average rate of change to determine the average speed of an object from a given distance–time graph**

As per its definition, the 'average rate of change' can describe many different ideas, but in motion, the average rate of change would describe the average speed of the object we are considering. We may determine the average speed of an object through what we call a distance-time graph.

A distance-time graph plots the distance an object travels in its path, with each point plotted in time. The graph below is what we call a non-linear distance-time graph.




The red graph describes the object's total distance travelled vs time, while the dashed-black line is what we would draw to find the average speed of the object over the first four seconds.

In simple words, we would draw a line between two points on a distance-time graph (with specific times that we want) and find its gradient - the gradient's value is the average speed over the interval.

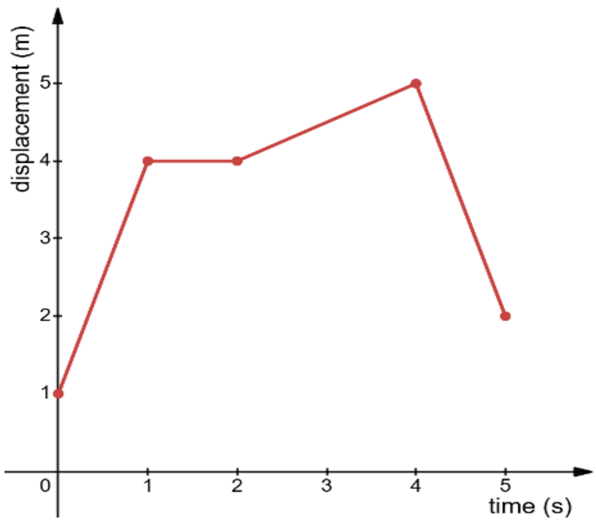
We can double-check this (or even confirm this when unsure) by checking our units.

# THEORY

| Gradient | Speed |
|---|---|
| | |

**EXAMPLE QUESTION**

**An object's distance from a given point is plotted over an interval of five seconds.**




**Find the following:**

a. Total distance travelled

**(1 mark)**

*STEP 1: Add distance travelled in each separate time interval*

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b. Overall Displacement

**(1 mark)**

*STEP 1: Subtract distances (away from given point) of first and final position*

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c. Average speed

**(1 mark)**

*STEP 1: Add distance travelled in each separate time interval*

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*STEP 2: Divide distance by time*

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# THEORY

d. Average velocity

**(1 mark)**

*STEP 1: Subtract distances (away from given point) of first and final position*

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*STEP 2: Divide displacement by time*

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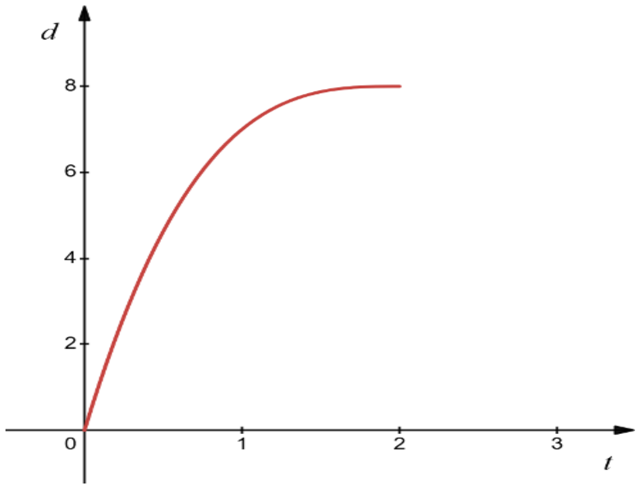
*PRACTICE QUESTIONS*

**1. Find the average speed of the following objects throughout their motion. Units are in metres and seconds.**

**(1 mark)**




a.

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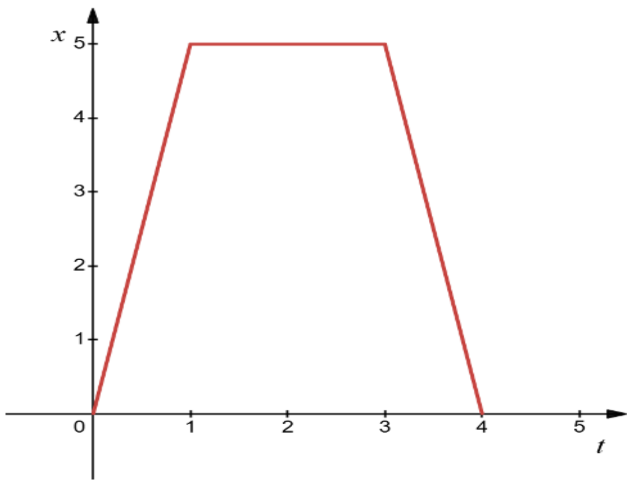
**(1 mark)**




b.

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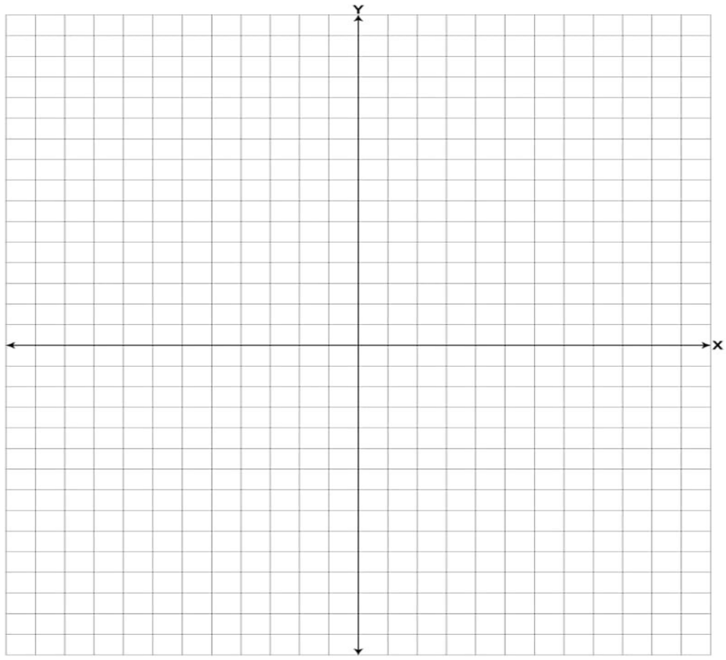
# THEORY

2. A car's displacement $x$ m from its starting point follows $x = t(3 - t)$, for $t > 0$. Find the car's average speed throughout the first 4 seconds.

(3 marks)




## Instantaneous Speed

**4.1.4 Describe the difference between the average speed of an object and its instantaneous speed**

**4.1.5 Determine that the instantaneous speed of an object at time $t$ can be approximated by the average speed between its position at time $t$ and its position some time later, and explain how this approximation can be improved**

"Average speed" and "instantaneous speed" are both rates of change and are similar concepts. They use the same units but take note that they are still two different ideas.

**Average speed** measures the ……………………………………………………

**Instantaneous speed** measures the ……………………………………………………

# THEORY

We've looked at how to find average speed, but finding instantaneous speed is a different case, and can be challenging if our object's speed is always changing.

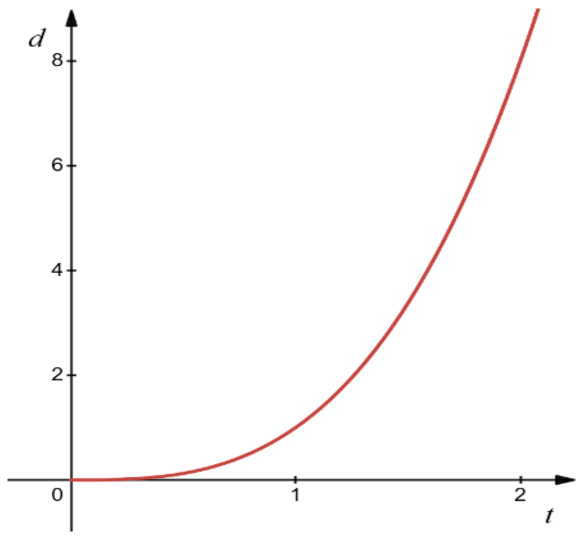
The word instantaneous tells us that it is an instant, a single point in time. The next closest thing to a single point in time would be an extremely small-time interval. This could be as big as 1 second, or as small as 0.1 seconds.

Calculating the speed over such a short period of time will be relatively close in value to the true instantaneous speed, as this is the next closest thing.

Let's try this with an example question, where we'll see that the smaller the time period is, the more accurate and closer we are to the instantaneous speed.

## EXAMPLE QUESTION

**The distance a car travels follows the function $d = t^3$ after $t$ seconds. It is known that at 1 second, the car's instantaneous speed is $3ms^{-1}$. Approximate the car's instantaneous speed at 1 second, with respect to the following time intervals:**




a. $0 - 2$ seconds

**(1 mark)**

*STEP 1: Find distance travelled*

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*STEP 2: Divide distance by time*

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# THEORY

b. $0.5 - 1.5$ seconds

**(1 mark)**

*STEP 1: Find distance travelled*

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*STEP 2: Divide distance by time*

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c. $0.9 - 1.1$ seconds

**(1 mark)**

*STEP 1: Find distance travelled*

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*STEP 2: Divide distance by time*

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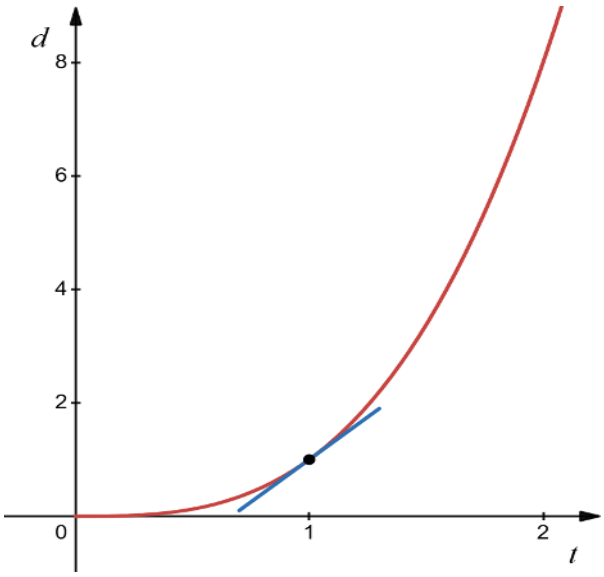
### 4.1.6 Relate the instantaneous speed of an object to the gradient of the tangent at that point on its distance–time graph

### 4.1.7 Estimate the instantaneous speed of an object from its distance–time graph

You may have noticed in the previous example question that as the time interval decreased (in other words, when the two time points were closer to each other), the straight line we'd normally draw connecting them became a better approximation of the slope of the curve at that moment.

This is no coincidence. When two time points coincide, the resulting line becomes the tangent to the curve at that single point in time, and its slope is the precise instantaneous velocity. This means that the resulting line has a gradient equal to the object's instantaneous speed at that point in time.

# THEORY




Remember that while instantaneous velocity has a magnitude equal to instantaneous speed, it also has a direction, as it's a vector quantity. This means we can also find instantaneous velocity from displacement-time graphs, but we need a direction as well.

Throughout the Advanced Maths syllabus, the functions we have looked at can be categorised into two types:

a. Non-linear Functions; and
b. Linear Functions.

The method for finding the instantaneous rate of change in each function differs. For non-linear functions, the method is a bit more complex and draws from the previous ideas of tangent gradients.

# THEORY

## Non-Linear Functions

**4.1.9 Recognise when modelling with a non-linear function that the rate of change is not constant and is represented by the gradient of the tangent to the curve at each point on the curve**

**4.1.10 Estimate the instantaneous rate of change of a non-linear function at a given point from a given graph of a practical situation**

To recall, non-linear functions are curves and hence do not have constant gradients.

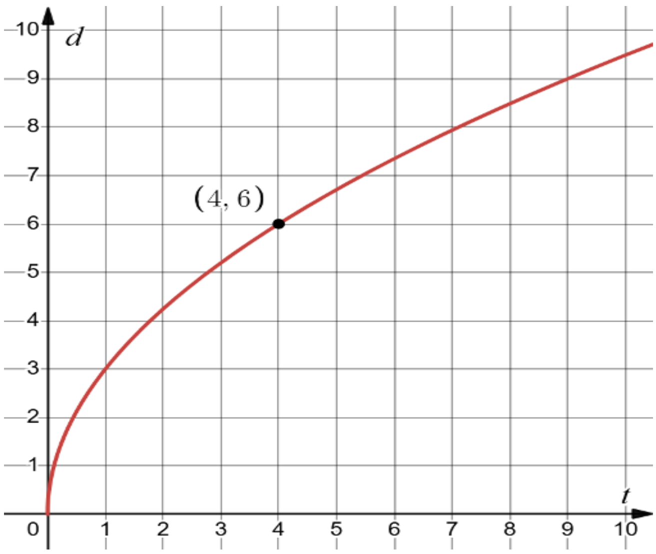
So, to estimate the instantaneous rate of change of an object at a certain point of time in these types of functions, we need to:

- Draw ...................................................................... by drawing a line perpendicular to the graph at that point; and
- Calculate ......................................................................by choosing two points on the tangent

This is the best method that we can use so far to get as close as we can to the true instantaneous rate of change. In the near future, we'll be faced with concepts called 'derivatives', which face extremely similar methodologies, but are even more accurate.

### EXAMPLE QUESTION

**The function below plots the distance $d$ km a car travels after $t$ minutes.**

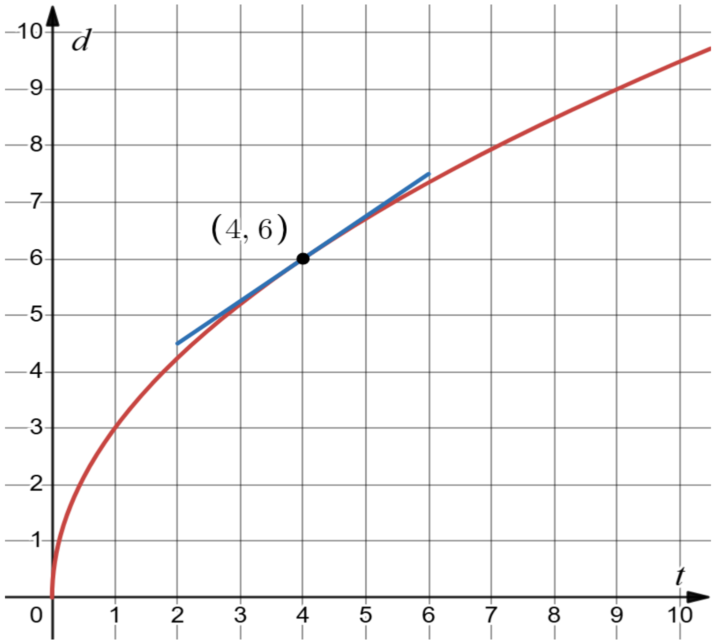



**Estimate the instantaneous speed of the car at 4 minutes.**

**(2 marks)**

# THEORY

*STEP 1: Draw tangent at point*




*STEP 2: Choose 2 points on tangent*

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*STEP 3: Find gradient of tangent*

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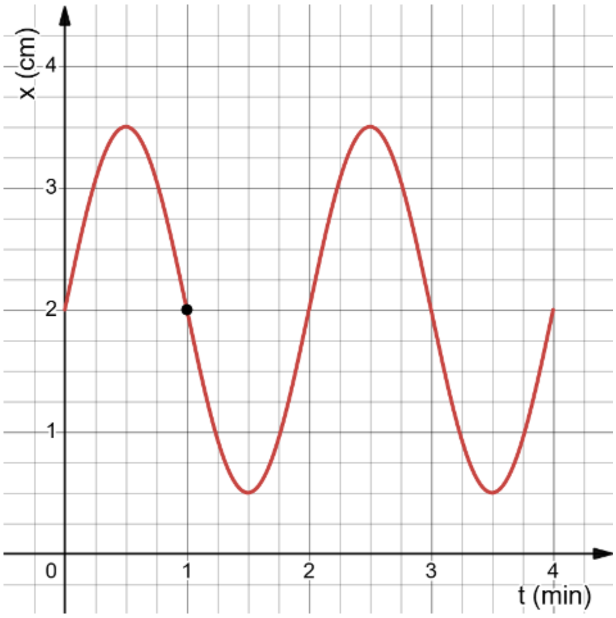
# THEORY

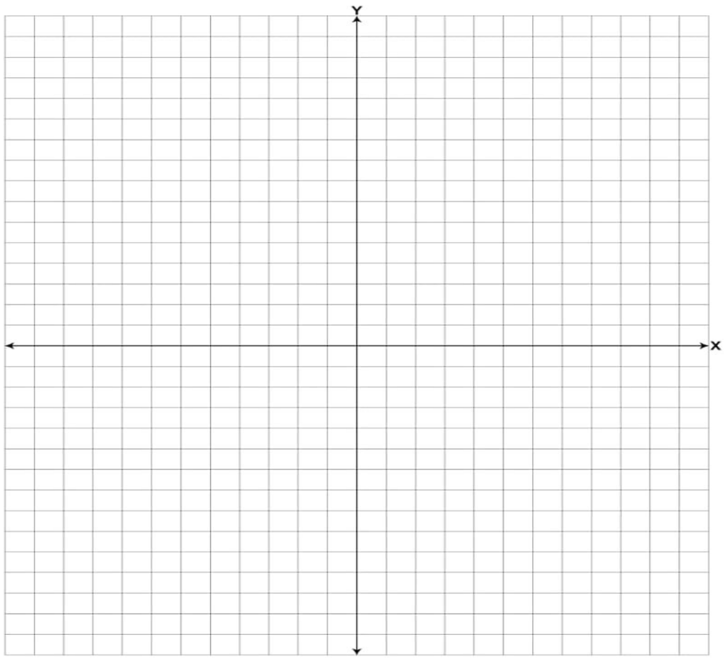
*PRACTICE QUESTIONS*

Estimate the instantaneous rates of change at each point for the following graphs.

(2 marks)

a.

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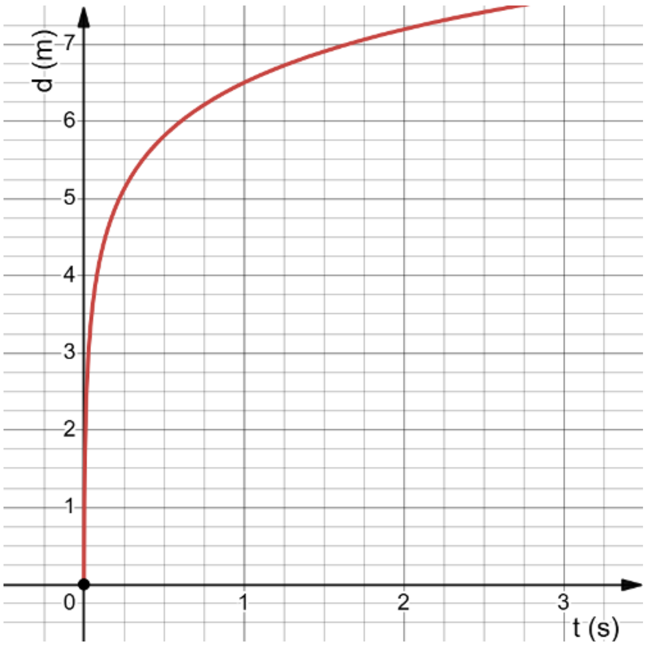
# THEORY

(2 marks)

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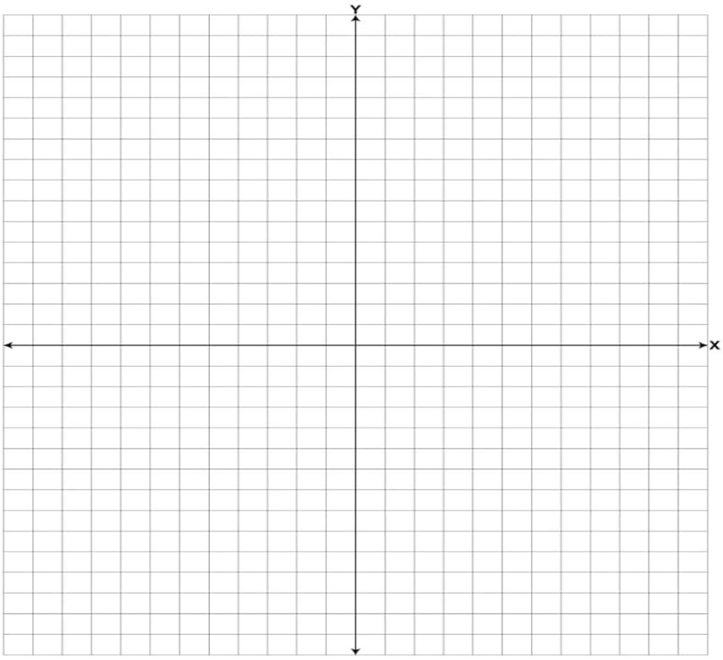


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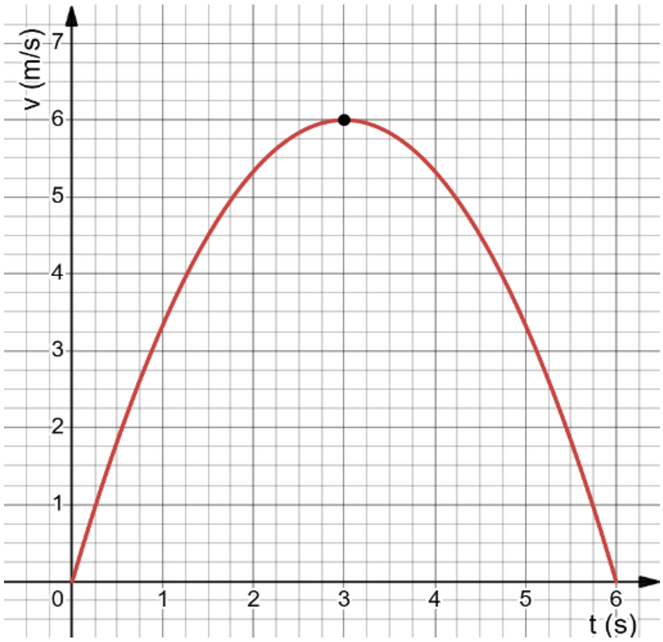
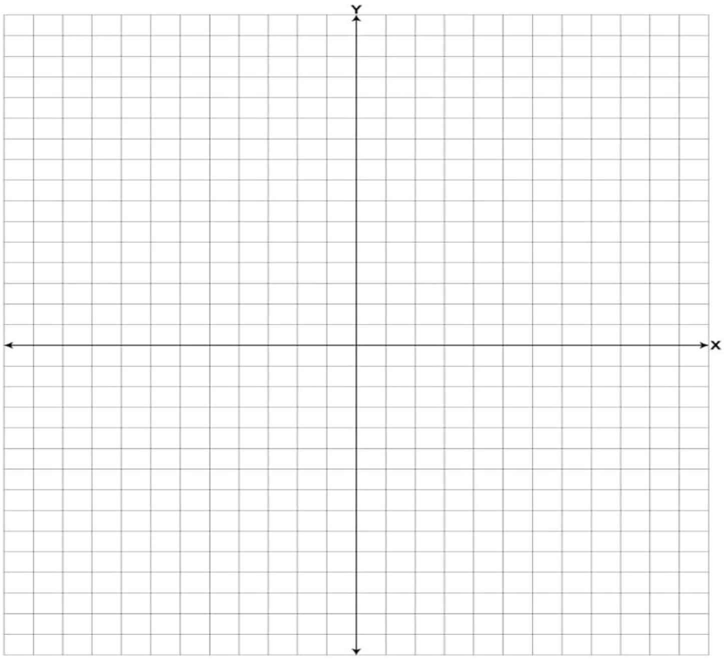
# THEORY

(2 marks)




c.

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# THEORY

## Linear Functions

**4.1.8 Recognise when modelling with a linear function that its gradient is the rate of change and determine the rate of change for linear functions in practical situations**

All linear functions are straight lines, and have two possible cases:

- $y = mx + b$
  - Contains a variable → degree of .....................................
- $y = b$
  - Contains a variable → degree of .....................................

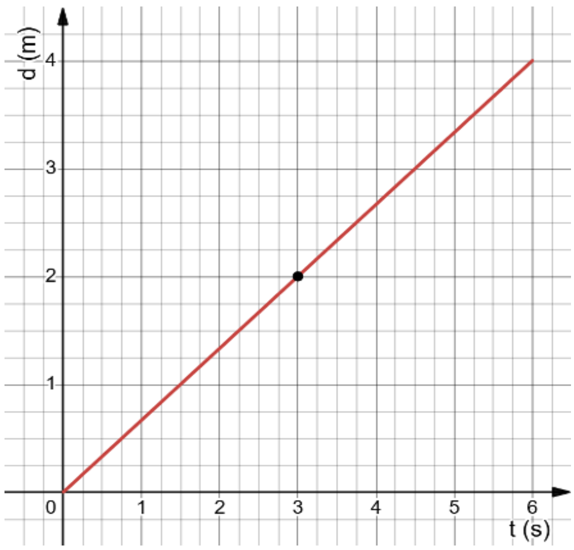
A straight line, in both cases, means the gradient is constant. In other words, the speed of the object is also constant and will not change.

So, to find the instantaneous rate of change for any linear function, we simply find the gradient between any two points on the function.

EXAMPLE QUESTION

**Find the instantaneous rate of change for the following function.**

**(1 mark)**




*STEP 1: Choose 2 points on graph*

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# THEORY

*STEP 2: Find gradient of function*

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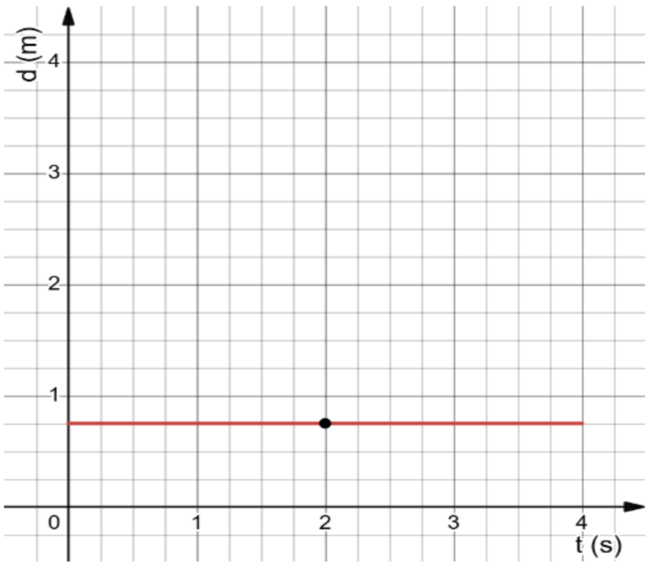
*PRACTICE QUESTIONS*

**Find the instantaneous rates of change for each of the following functions.**

**(1 mark)**




a.

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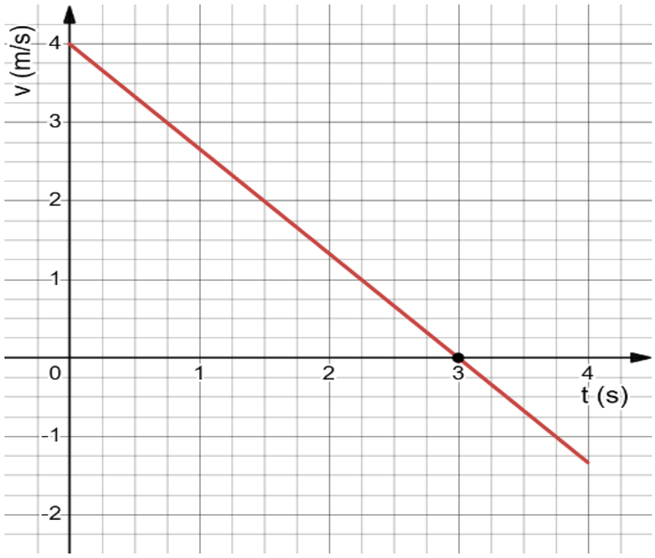
(1 mark)




b.

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## Let's Wrap Up!

Today, we learnt about rates of change and how they relate to an object's motion.

We were able to differentiate that concepts such as distance and speed don't have a direction, whilst their counterparts, displacement and speed, as well as acceleration, are vector quantities and therefore have a direction. Then, we looked at distance-time graphs and were able to find average speeds, as well as ways to approximate an object's instantaneous speed. We learnt that these same methods could be applied to finding velocity and acceleration, as long as we were dealing with the right graphs.

Next week, we'll be looking at a concept that builds a little bit off what we learnt today: differentiation! This is a concept used heavily in the syllabus of maths that introduces new ways for more accurate calculations, such as finding the rates of change at a single point. Over the next few weeks, we'll learn how it can additionally help us analyse and sketch both simple and more complex functions, and countless more of its uses in other topics of maths.

# FORMULA SHEET

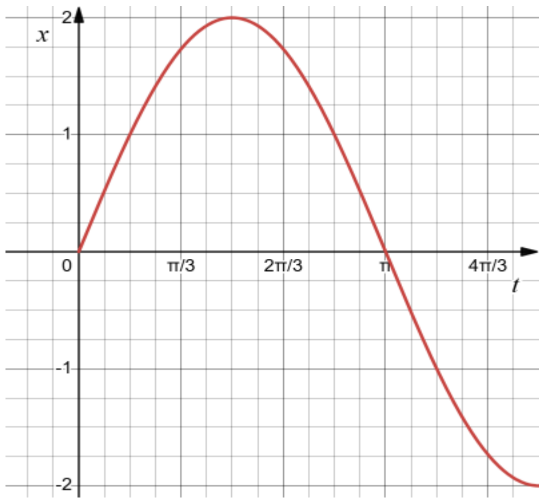
$$\frac{\Delta y}{\Delta x} = \frac{change\ in\ y}{change\ in\ x} = \frac{f(b) - f(a)}{b - a}$$

# HOMEWORK

1. **Which of the following is true about a linear function's rate of change?**

   a. Always positive
   b. Always negative
   c. It is constant
   d. It may be variable

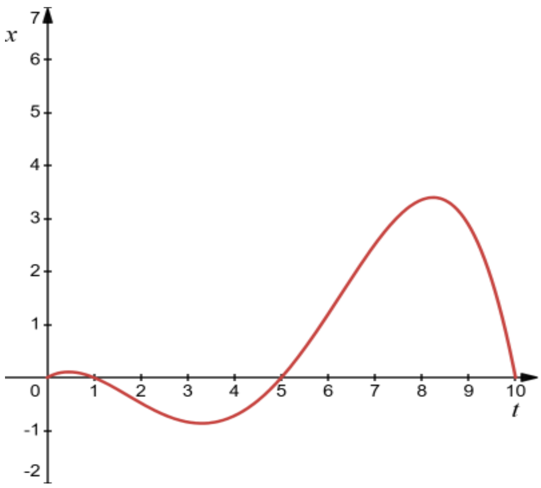
2. **A pendulum's displacement from its equilibrium position $x$ m after $t$ seconds follows the function $x = 2\sin t$. Its approximate velocity at $t = \frac{5\pi}{3}$ seconds is:**




**(1 mark)**

   a. $0.5\text{ms}^{-1}$
   b. $-0.5\text{ms}^{-1}$
   c. $1\text{ms}^{-1}$
   d. $-1\text{ms}^{-1}$

3. **A car starts from rest at point O and moves off. Its displacement is depicted in the graph below.**




**Which of the following is true about the car's motion after it started moving?**
**(1 mark)**

   a. It changed direction 4 times
   b. It reached the same velocity 3 times
   c. Its greatest speed was at $t = 5$ minutes
   d. One place was crossed at least 4 times

# HOMEWORK

4. This is a 2-part question. Use this stimulus for questions 4 and 5.

   CJ runs with a velocity $v$ $ms^{-1}$, where $v = 4e^{-t^2} - 2$ after $t$ seconds. Positive values are in the Eastern direction, while negative values are in the Western direction. It is known that CJ only stops once throughout his run.

   What do we know about CJ's acceleration when he is stationary?

   **(1 mark)**

   a. CJ is accelerating towards the Eastern direction
   b. CJ is accelerating towards the Western direction
   c. It is 0
   d. His acceleration doesn't change

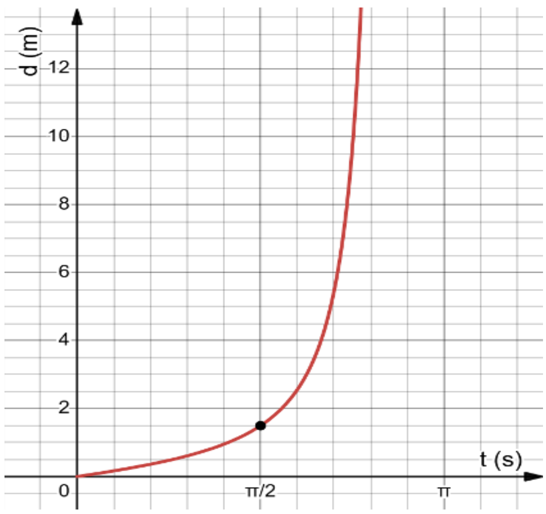
5. What is his average acceleration over the first ten seconds?

   **(1 mark)**

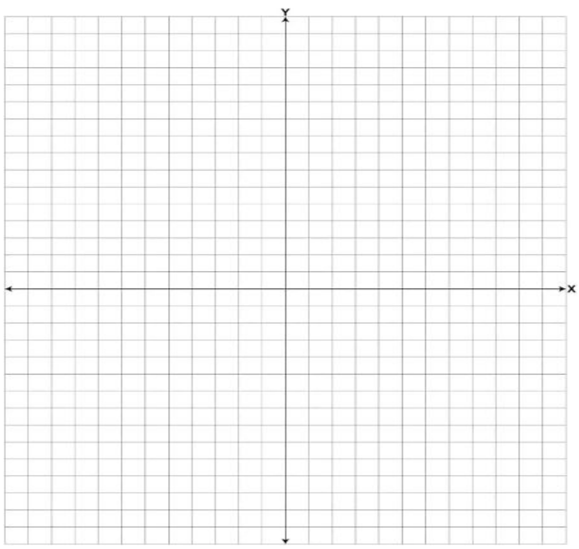
   a. $-0.4ms^{-1}$
   b. $-0.8ms^{-1}$
   c. $4ms^{-1}$
   d. $8ms^{-1}$

6. Find the approximate rates of change for each of the following graphs.

   **(2 marks)**




a.

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# HOMEWORK

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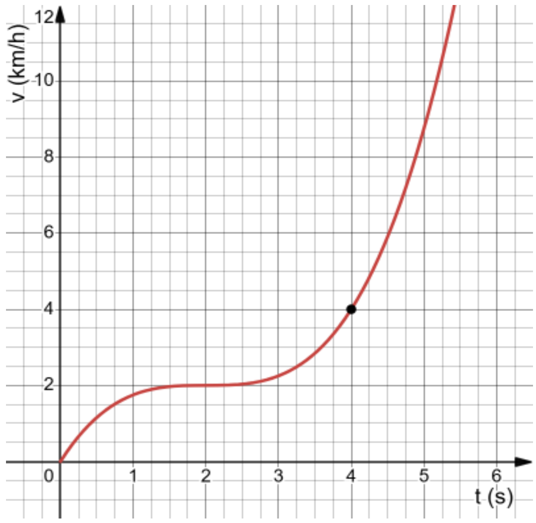


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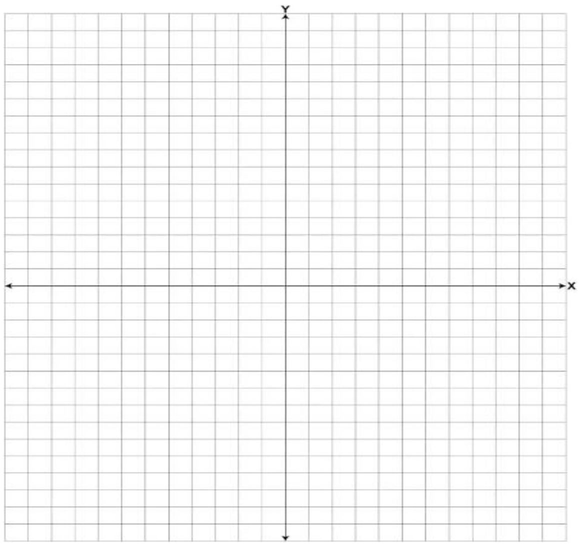
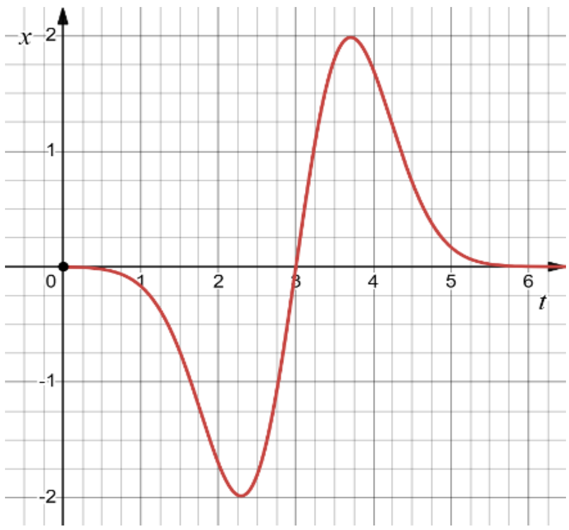



7. The displacement of an electron x m from its equilibrium position after t ms is modelled by the following graph.



# HOMEWORK

a. Calculate the following in the interval $t = 0$ to $t = 5$:

i. Total Displacement

**(1 mark)**

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ii. Average Speed

**(1 mark)**

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b. Compare the accelerations of the electron at $t = 2.375$ and $t = 5.25$.

**(3 marks)**

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c. Given that the electron only crosses its equilibrium position once after it moves off, describe the final displacement of the electron.

**(2 marks)**

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**8. An object's velocity $v$ ms$^{-1}$ is given by $v = t^2 - 4t + 6$ after $t$ seconds. Find the average acceleration between $t = 1$ and $t = 3$.**

**(2 marks)**

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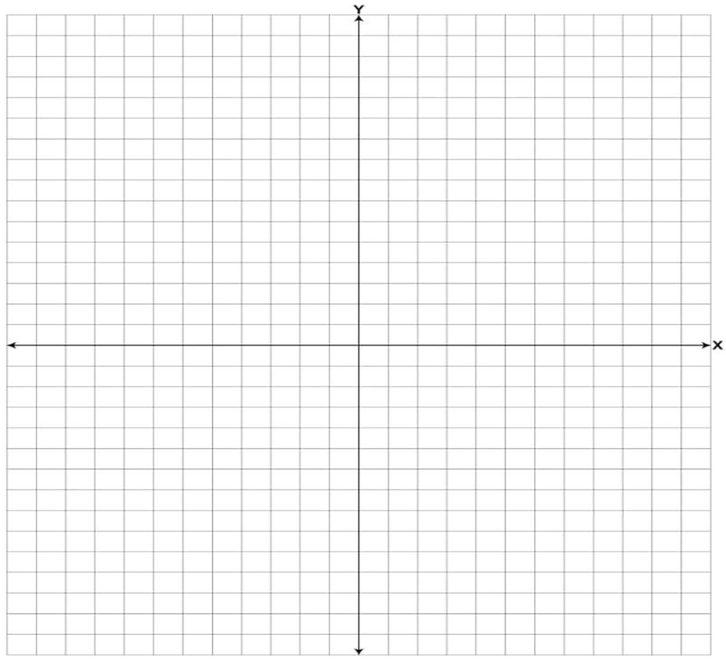
# HOMEWORK

**9. Billy runs up a hill, but trips over and rolls down the hill. where his height above the ground h m follows the piecewise function**

$$f(x) = \begin{array}{l} 2t \text{ if } 0 < t < 2 \\ (t - 4)^2 \text{ if } 2 < t < 4 \\ 2(t - 4)^2 \text{ if } 4 < t < 6 \end{array}$$

a. Sketch the piecewise function

**(2 marks)**




b. Calculate the average velocities of each 2-second interval

**(3 marks)**

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c. It is now known that Billy's velocity follows the piecewise function:

$$f(x) = \{ \begin{array}{l} 2 \text{ if } 0 < t < 2 \\ 2t - 8 \text{ if } 2 < t < 4 \\ 4t - 16 \text{ if } 4 < t < 6 \end{array}$$

Find the greatest magnitude of Billy's instantaneous acceleration and when this occurs.

**(2 marks)**

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# HOMEWORK

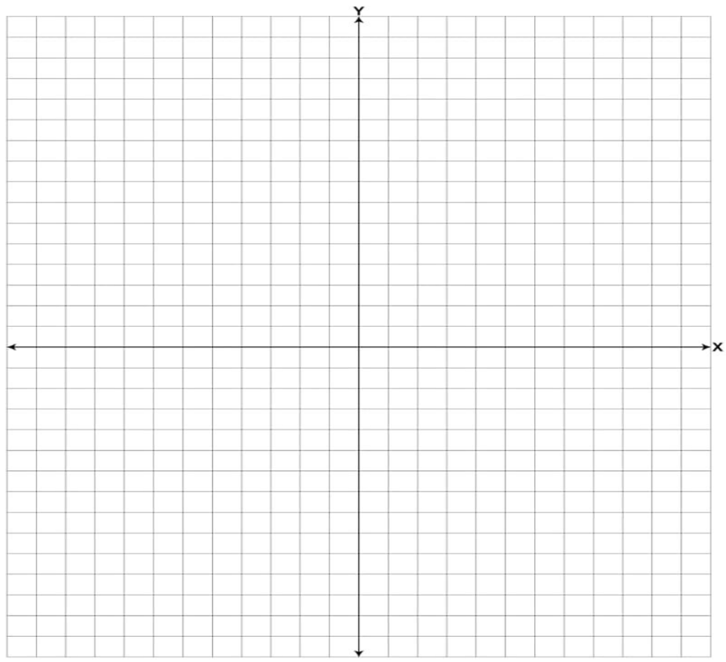
10. **Romeo drives from his home straight to the gym, averaging $45\text{km/h}$ for $20$ minutes. He stays at the gym for 20 minutes and drives straight to the shops at $30\text{km/h}$ for $10$ minutes.**


a. Calculate his average velocity.

**(2 marks)**

b. It is known that Romeo spent 20 minutes at the gym. Create a fully labelled distance-time graph of Romeo's journey.

**(2 marks)**




c. Calculate his average speed over the whole journey.

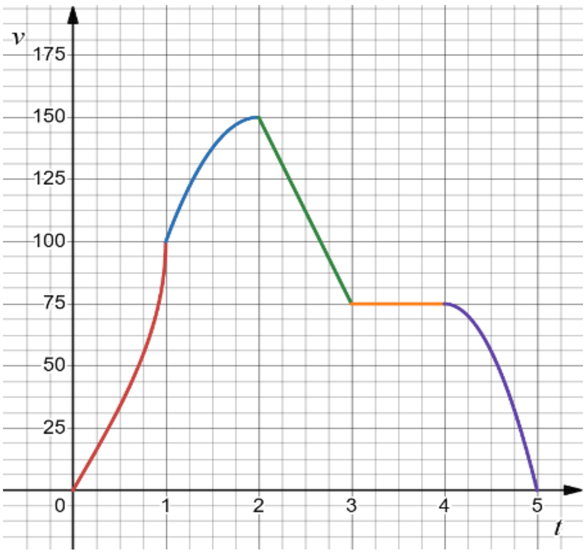
**(1 mark)**

d. Hence, explain the differences between total distance and displacement travelled, and average speed and velocity.

**(2 marks)**

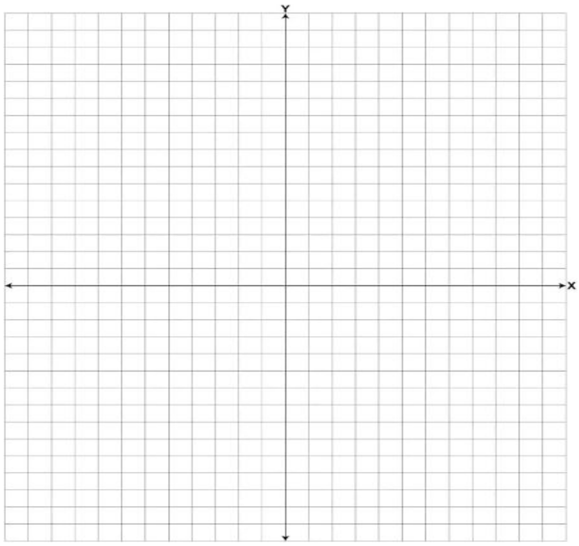
# HOMEWORK

11. **Nathan and his friends decide to go on a road trip, which takes 5 hours. The velocity $v\,\mathrm{km/h}$ of their car is plotted on the graph below.**




a. Approximate the acceleration of the car at 1.5 hours.

**(2 marks)**




b. Identify when the instantaneous speed is highest.

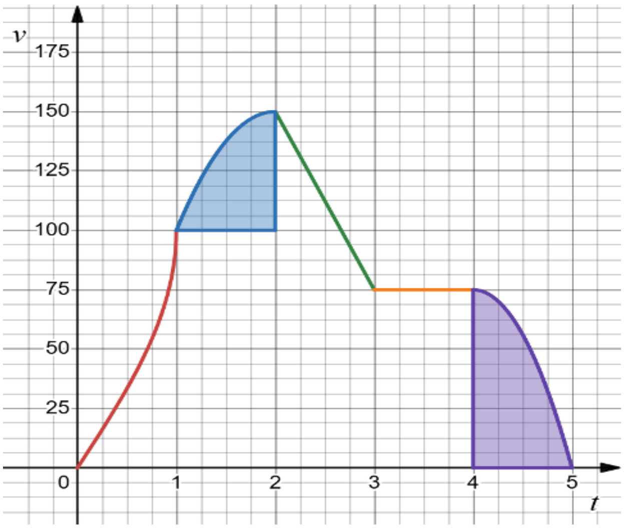
**(1 mark)**

c. Imagine a rectangle with the endpoints of the red portion of the journey used as its corners. Recalling that $\text{distance} = \text{speed} \times \text{time}$, find the distance covered in the first hour, given the red curve halves the rectangle.

**(1 mark)**

d. Hence, given the blue and purple regions have an area of $\frac{100}{3}$ $u^2$ and $50\ u^2$ respectively, calculate the average speed of the car throughout the whole journey.

**(3 marks)**

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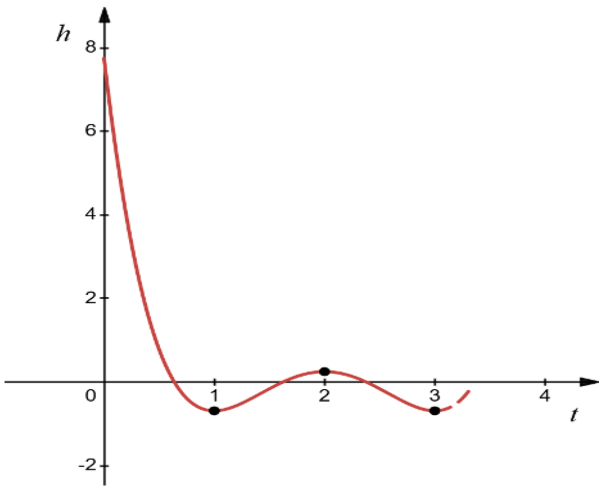
**12. A rollercoaster's displacement $h$ m above the ground after $t$ seconds is modelled by the function below. The rollercoaster stops after 3 seconds.**




**Identify when the instantaneous speed is 0 and explain how this is graphically shown.**

**(3 marks)**

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# HOMEWORK

13. **Two trains are running on parallel tracks, side by side. The displacement $x$ km from their starting point after $t$ hours is as follows:**

**Train A:**

$$x_A = 2t$$

**Train B:**

$x_B = (t-2)^2$ $(t > 2)$

**They manage to reach their destination at the same time, $t = \alpha$ hours.**

a. Over the whole journey of $t = \alpha$ hours, identify which train has a greater:

i. Displacement

**(1 mark)**

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ii. Average Velocity

**(1 mark)**

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iii. Average Acceleration

**(1 mark)**

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b. Identify when Train B's instantaneous velocity is highest throughout the journey.

**(1 mark)**

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c. Let it be known that Train A reaches its velocity instantly, whereas Train B starts from rest and accelerates. Is there an instance from $t = 0 \to \alpha$ when they have the same velocity? Explain why or why not.

**(2 marks)**

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d. Find $\alpha$.

**(2 marks)**

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# HOMEWORK

## CHALLENGE QUESTION

1. **An object's path is plotted on a displacement-time graph, travelling in a forward direction. However, the tangents drawn at each point become progressively less steep.**

    a. Describe what is happening to the object's motion.

    **(1 mark)**

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    b. What are the directions of the three vector quantities?

    **(1 mark)**

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    c. Hence, what conclusions can we make about an object in motion from the product $av$?

    **(2 marks)**

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elevate