elevate

YEAR 12 MATHS ADVANCED

MODULE 7

LESSON ONE

THEORY

Applications of Integration

In the past few weeks, we've been looking at how to integrate different types of functions. These have included simple functions such as polynomial and logarithmic functions, as well as more complex composite functions.

Today, we'll put this to use and explore how integration can be used in different areas of maths; more specifically, we'll be understanding how the integral of a function within an interval is related to the area between the graph and the x or y-axis.

We'll be covering:

- Using Definite Integrals
- Finding Areas with Integration
- Properties of Definite Integrals
- Symmetry and Transformations
- Integration in the Real World

Using Definite Integrals

11.8.1 Apply $\int_a^b f(x)\,dx = F(b) - F(a)$, where $F(x)$ is a primitive of $f(x)$, to calculate definite integrals and solve related theoretical problems involving functions within the scope of the Mathematics Advanced course

Let's do a quick recap of definite integrals and the Fundamental Theorem of Calculus.

Contrary to indefinite integrals, definite integrals have ....

This means they have one value, which is equivalent to the value of the specified .... between ....

The Fundamental Theorem of Calculus is defined as:

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Therefore, we can use this theorem to solve definite integrals for their value.

Let's see how we can put this to use, by attempting to calculate the distance, the primitive of velocity, a car has travelled, given its velocity.

THEORY

EXAMPLE QUESTION

A car travels with a velocity V ms^{-1} defined by the equation $\boldsymbol{V} = 2\boldsymbol{t}$ at any time t seconds. How far has the car travelled, starting from rest, over a period of 60 seconds?

STEP 1: Define what we're solving and how we're solving

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STEP 2: Integrate velocity

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STEP 3: Substitute our bounds (0 and 60) and simplify

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PRACTICE QUESTIONS

1. Jack plants his beanstalk outside and goes to sleep. The height h m of his beanstalk grows at a rate given by $\frac{dh}{dt} = 2e^{2t}$ after t hours. Given that he can only sleep for 3 hours, how tall has his beanstalk grown during his sleep?

(2 marks)

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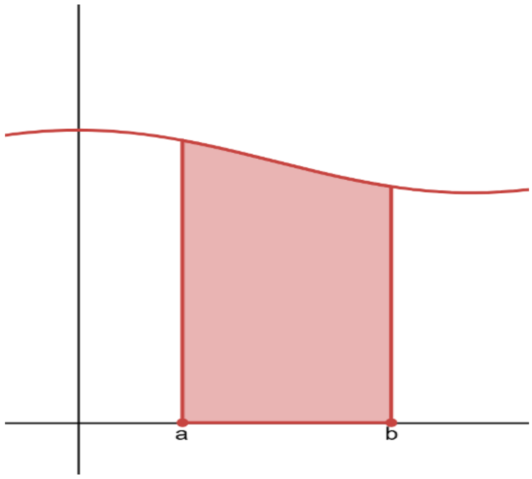
THEORY
Finding Areas with Integration

11.8.2 Describe, in the case where $f(x) \geq 0$ for all values of x in the interval $a \leq x \leq b$, the area bounded by the graph of the continuous function $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$, as $\int_a^b f(x)dx$

We can recall from our previous lessons that when we integrate a curve between two x-values, we're actually solving for the ....

To do this, we would find a definite integral, substitute in our wanted boundaries, and solve for the value.

For example, let's consider the function $f(x)$ and its graph below.



To find the area under the curve from a to b, which is the shaded area, we would evaluate the integral:

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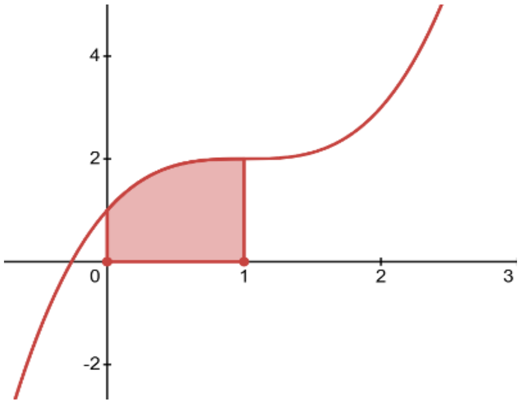
THEORY

EXAMPLE QUESTION

Given the function $f(x) = (x-1)^3 + 2$ and its graph below, find the area of the shaded region.

(2 marks)



STEP 1: Sub function and bounds into $A = \int_a^b f(x)\,dx$

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STEP 2: Integrate f(x)

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STEP 3: Substitute bounds and simplify to find area

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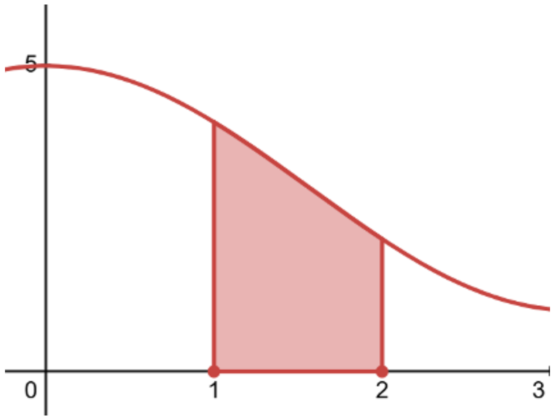
THEORY

PRACTICE QUESTIONS

1. Given the graph of $y = 2cosx + 3$, find the area of the shaded region.



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2. Find the area bounded by the graph $y = e^{(2x-3)}$, the x-axis and horizontal lines x = 0.5 and x = 1.5.

(2 marks)

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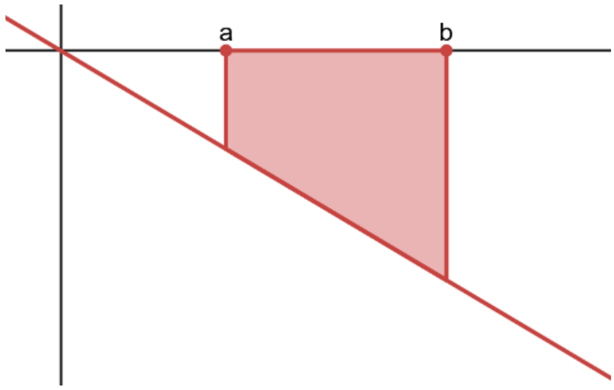
11.8.3 Recognise, in the case where $f(x) \leq 0$ for all values of x in the interval $a \leq x \leq b$, the area bounded by the graph of the continuous function $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$, as $\left|\int_a^b f(x)\,dx\right|$ or $-\int_a^b f(x)\,dx$

However, not all graphs will be above the x-axis. You may have noticed that the last question produced a negative instead of a positive value for the definite integral. Areas can be negative, so what does this mean?

THEORY

This means that within the interval, the area between the graph and the x-axis is actually under, and not above the x-axis. This results in a negative value rather than a positive.

Some questions, like that last question, will ask us to find the area between the curve and the x-axis, but won't inform us of whether the area is above or below the x-axis. This could be as simple as asking us to find the area 'under' a linear function, exemplified by the graph below.



To avoid obtaining any negative values for our areas, we can simply use an absolute value on the definite integral, or apply a negative to the entire integral as such:

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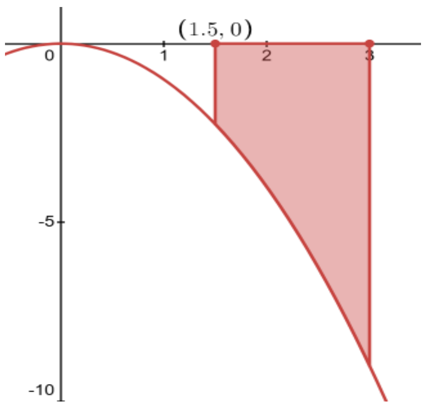
EXAMPLE QUESTION

Given the graph of $y = -x^2$, find the area beneath the curve between x = 1.5 and x = 3.



STEP 1: Sub function and bounds into $A = |\int_a^b f(x)\, dx|$

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THEORY

STEP 2: Integrate f(x)

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STEP 3: Substitute bounds and simplify to find area

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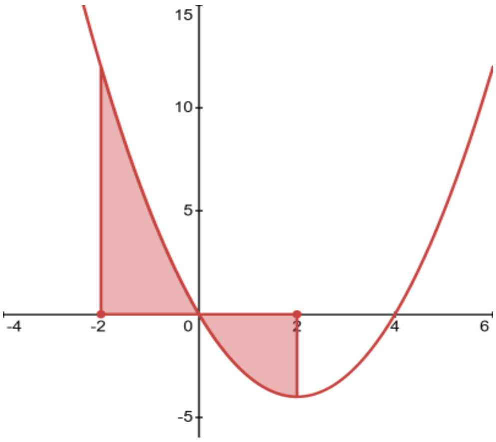
PRACTICE QUESTIONS

1. **Given the graph of $y = (x-2)^2 - 4$, calculate the area between the curve and the x-axis using the boundaries x = - 2 and x = 2.**



(1 mark)

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THEORY

11.8.4 Conclude, for a continuous function $y = f(x)$ on the interval $a \leq x \leq b$, that $\int_a^b f(x)\,dx =$ (area of regions between curve and x-axis lying above the x-axis) − (area of regions between curve and the x-axis lying below the x-axis)

11.8.8 Recognise and use the result, where $f(x)$ is continuous on the interval $a \leq x \leq c$, $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$ for all c such that $a \leq b \leq c$

In the last practice question, we were faced with a graph that had a region both above and below the x-axis. However, we couldn't just integrate the whole region as normal to obtain the total shaded area. This occurs due to the property of definite integrals:

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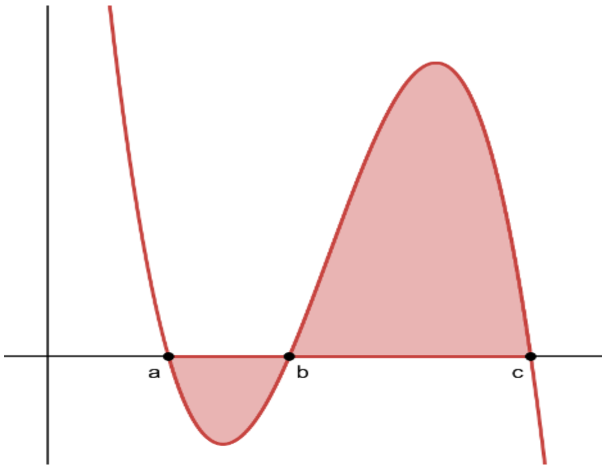
Let's consider an unknown graph f(x) with x-intercepts $(a, 0)$, $(b, 0)$ and $(c, 0)$.



The property above would allow us to separate the integrals to make it easier to calculate.

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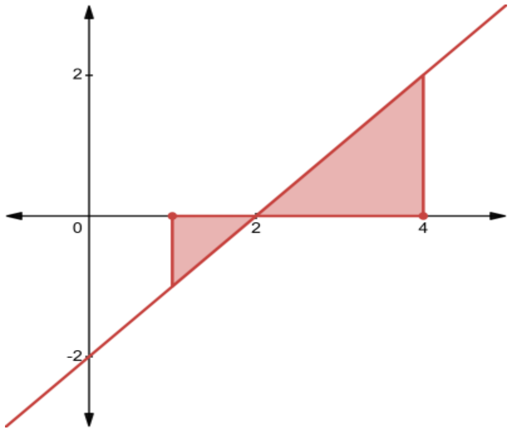
However from the graph we can deduce that $\int_a^b f(x)dx$ will be negative as this part of the graph is below the x-axis. As areas can only be positive, the whole integral will give us a subtraction of the areas rather than an addition of their areas.

So, if we are looking to find the 'area' specifically - and the function crosses the x-axis in between the two boundaries, we **must find the areas of the two regions separately** and add them together.

THEORY

EXAMPLE QUESTION

Consider the graph of $y = x + 2$ and the bounds $x = 2$ and $x = 4$. Find $I = \int_1^4 x - 2\, dx$ by:



a. Using the area formula from above

STEP 1: Find the area of the 2 separate regions

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STEP 2: Substitute value of areas into formula

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b. Integrating directly

STEP 1: Sub function and bounds into $A = \int_a^b f(x)\, dx$

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STEP 2: Integrate f(x)

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STEP 3: Substitute bounds and simplify to find area

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THEORY

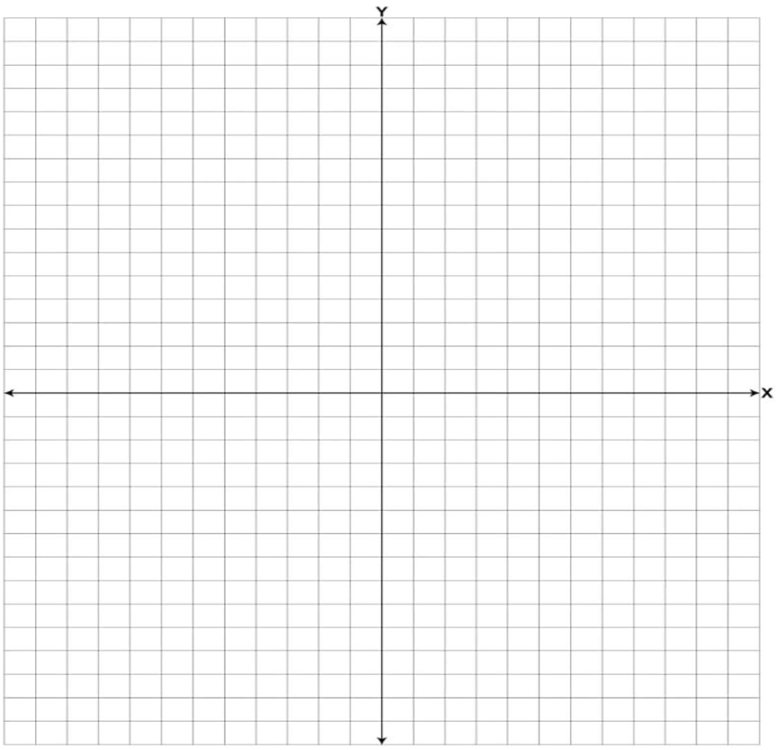
In this particular case, you do need to take the absolute values and the areas, because you're strictly looking for the value of that integral, and not just 'areas'. The area under the curve in this case, is negative and that negative sign, is significant.

PRACTICE QUESTIONS

By sketching the graph of $y = x^2 + 4x + 3$ or otherwise, find the value of $I = \int_{-5}^{0} (x + 3)(x + 1)\, dx$ by:

a. Using the area formula, given that:
   - $\left| \int_{-5}^{-3} (x + 3)(x + 1)\, dx \right| = \frac{20}{3}$
   - $\left| \int_{-3}^{-1} (x + 3)(x + 1)\, dx \right| = \frac{4}{3}$
   - $\left| \int_{-1}^{0} (x + 3)(x + 1)\, dx \right| = \frac{4}{3}$

(2 marks)



b. Integrating directly

(2 marks)

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The property of definite integrals, $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$ is useful because it allows us to:

1. Split an integral

- To make a longer integral easier to solve

$\int_a^c f(x)dx \;\rightarrow\; \int_a^b f(x)dx + \int_b^c f(x)dx$

2. Add two integrals together

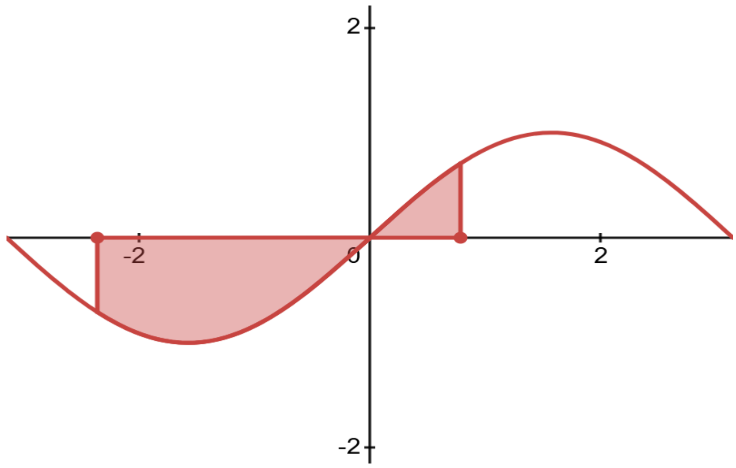
- If the bounds line up, the integral can be simplified and then solved

$\int_a^b f(x)dx + \int_b^c f(x)dx \;\rightarrow\; \int_a^c f(x)dx$

These will be especially helpful in integrating functions we haven't seen before, in which we can use ideas such as trigonometric identities to turn them into functions we have integrated before e.g. $sin^2x = \frac{1-cos2x}{2}$.

EXAMPLE QUESTION

1. **Find the area bounded by the equation $y = sinx$, the x-axis and the lines $x = -\frac{3\pi}{4}$ and $x = \frac{\pi}{4}$.**



THEORY

STEP 1: Define what total area is equal to

STEP 2: Sub function and bounds into $\boldsymbol{A = \int_a^b f(x)\,dx}$

STEP 3: Integrate f(x)

STEP 4: Substitute bounds and simplify to find area

STEP 5: Repeat for second region

STEP 6: Add areas together

PRACTICE QUESTIONS

1. **Given the function $f(x) = 2x^3 - x^2 - 3x$, find the area between the curve and x-intercepts.**

(4 marks)

THEORY

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11.8.9 Define and use the result $\int_a^b f(x)dx = -\int_b^a f(x)dx$, where $f(x)$ is continuous on the interval $a \leq x \leq b$

Another property we can use that utilises sign switching is:

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This property will be useful for solving certain integrals that require you to add integrals together if the bounds are the same, which makes it much easier to solve.

We can simply prove this using the Fundamental Theorem of Calculus. Recall the equation:

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This is the LHS of the equation, so let's expand the RHS of the equation to see if they're equal.

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PRACTICE QUESTIONS

1. **Find the value of $\int_0^{\pi}(tanx + 1)^2\ dx + 2\int_{\pi}^{0} tan\ x\ dx$**

(2 marks)

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THEORY

11.8.5 Use definite integrals to solve problems involving the areas of regions bounded by the graph of the continuous function $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$, in cases where $f(x) \geq 0$ throughout $a \leq x \leq b$, $f(x) \leq 0$ throughout $a \leq x \leq b$ or where $f(x)$ changes sign in the interval $a \leq x \leq b$, with or without the graph provided

To summarise, there are three separate cases to be conscious of trying to find the area between the graph and the x-axis trying to find the area:

1. $f(x) \geq 0$ throughout $a \leq x \leq b$

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2. $f(x) \leq 0$ throughout $a \leq x \leq b$

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3. $f(x)$ changes sign in the interval $a \leq x \leq b$

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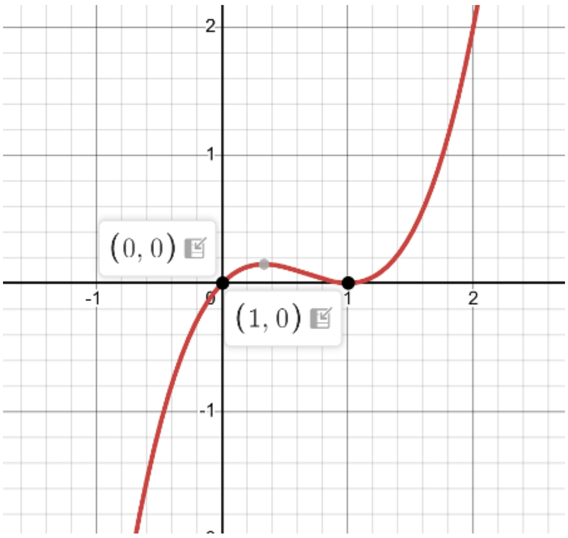
EXAMPLE QUESTION

Find the area bounded by the graph $y = x^3 - 2x^2 + x$, the x-axis and the lines x=1 and x = -1.

STEP 1: Consider a sketch of the graph

For these scenarios, it is best to have a visual representation of the graph so we know what we are dealing with.



THEORY

STEP 2: Find the intercepts with the x-axis

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STEP 3: Form the integrals to evaluate

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STEP 4: Evaluate the integrals

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STEP 5: Sum up the areas

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11.8.6 Use definite integrals to solve problems involving the areas of regions bounded by the graph of the continuous function $y = f(x)$, the y-axis and the lines $y = a$ and $y = b$, in cases where $x \geq 0$ throughout $a \leq y \leq b$, $x \leq 0$ throughout $a \leq y \leq b$ or where x changes sign in the interval $a \leq y \leq b$ with or without the graph provided

While we were able to solve the last few practice questions by subtracting the area of a triangle or a square from the area of the integral, another method to keep in mind is to find the area between a curve and the y-axis directly.

If we're given a function in terms of x, we can simply rearrange it into a function in terms of y, and then integrate with respect to y.

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Similar to what we have seen with integrating with respect to x, we can have 3 cases:

1. $fy \geq 0$ throughout $a \leq y \leq b$

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2. $fy \leq 0$ throughout $a \leq y \leq b$

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3. $f(y)$ changes sign in the interval $a \leq y \leq b$

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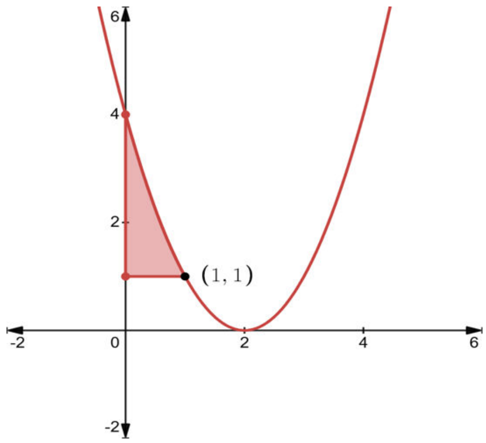
THEORY

EXAMPLE QUESTION

Find the area between the curve and the y-axis for each equation and its respective bounds:

a. $y = x^2 - 4x + 4$ with the bounds $y = 1$ and $y = 4$

STEP 1: Sketch function



STEP 2: Find equation in terms of y

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STEP 3: Sub function and bounds into $A = \int_c^d f(y)\, dy$

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STEP 4: Integrate f(y)

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STEP 5: Substitute bounds and simplify to find area

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THEORY

Area Between Two Curves

Let's say we have two curves, $f(x)$ and $g(x)$, where $f(x) \geq g(x)$ within the interval $a \leq x \leq b$. If we were to find the area between these two curves, we would separately find the area between each curve and the x-axis, and subtract from each other. Intuitively, this can be written down as the subtraction of the two integrals, resulting in the formula:

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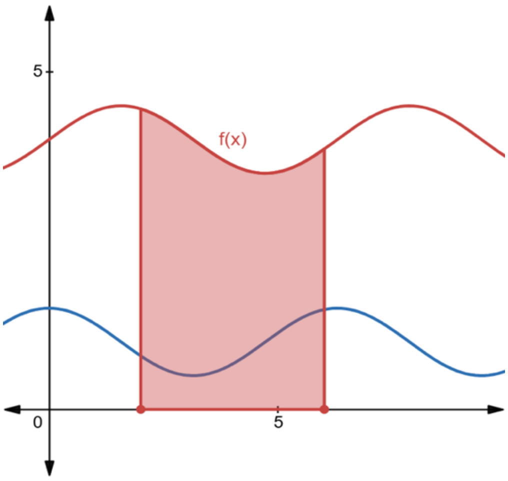

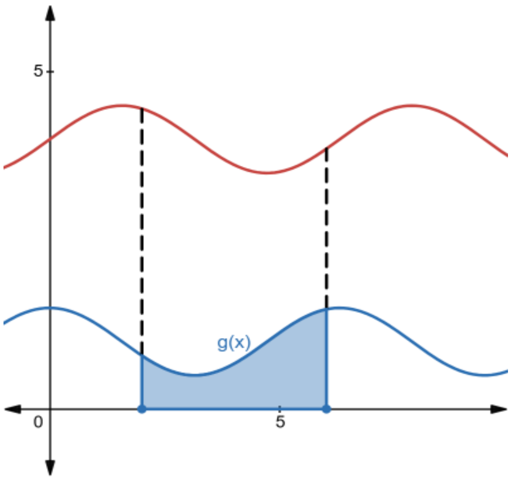

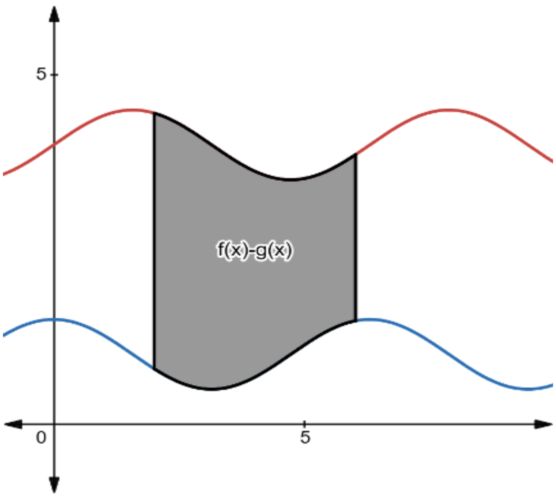

The same logic applies to finding the area with respect to the y-axis, where $f(y) \geq g(y)$ within the interval $c \leq y \leq d$.

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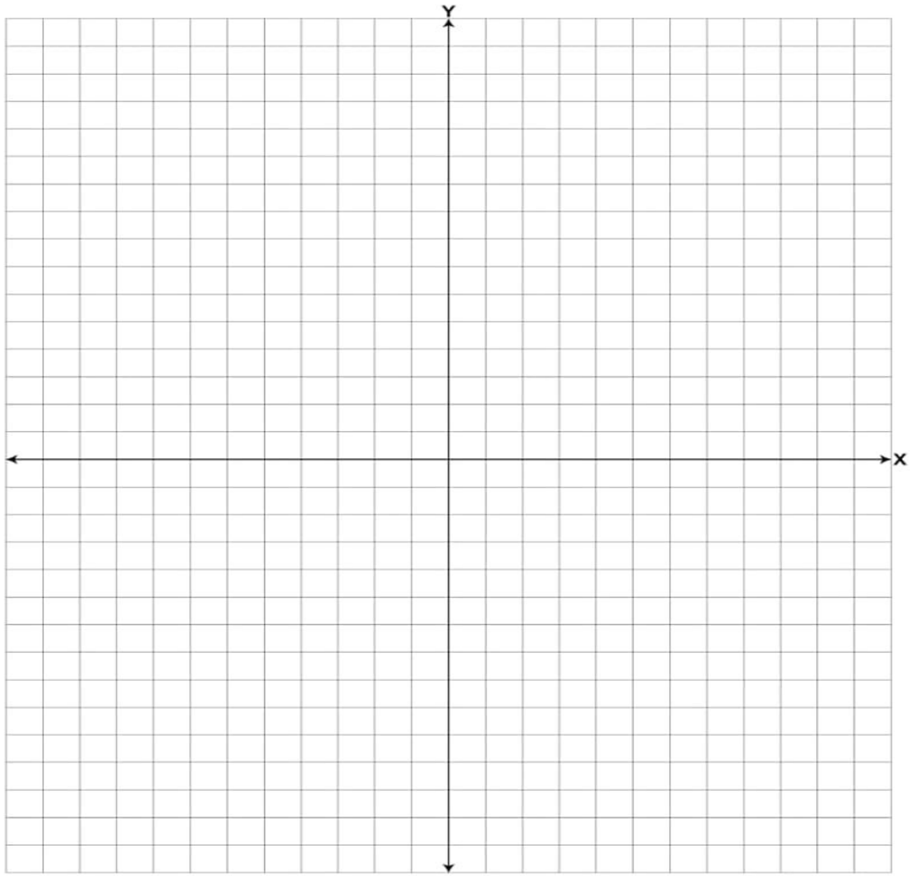
THEORY

EXAMPLE QUESTION

Find the area between the graphs $y = x + 9$ and $y = 9 - x^2$ in the interval $0 \leq x \leq 3$.

STEP 1: Sketch the curves + find any points of intersections



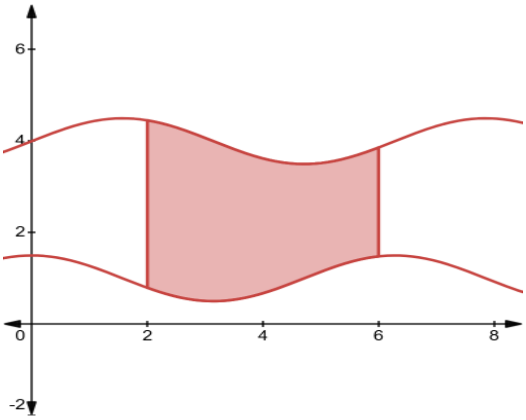
STEP 2: Write out formula for area (ensure to split regions at POI)

STEP 3: Integrate to find area

THEORY

PRACTICE QUESTIONS

2. **Given the graph of $y = \frac{sinx}{2} + 4$ and $y = cos^2(\frac{x}{2}) + \frac{1}{2}$, find the shaded area.**



(2 marks)

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11.8.7 Use the fact that the graphs of $y = a^x$ and $y = log_a x$ are reflections of each other in the line $y = x$ to solve problems involving areas between the x-axis or y-axis and a curve involving either an exponential or logarithmic function

We've managed to go through all the different types of finding areas, such as between a curve and the x or y-axis and between two curves.

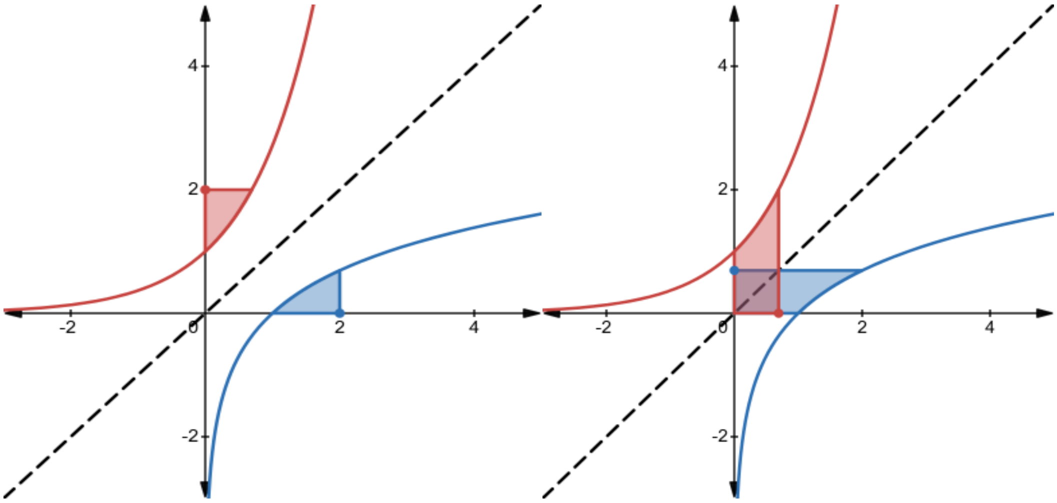
However, we may run into some questions that are a little trickier because they're more abstract or rely on geometric ideas. This may include:

- Strangely orientated regions
- Functions like lnx that we cannot integrate directly

A useful tip is to consider the inverse of a function.

When we graph a function and its inverse, such as the graphs $y = e^x$ and $y = lnx$, we can see something very important. That is, the area between a function and the x-axis is equal to the area between the inverse and the y-axis (with the bounds adjusted).

THEORY



Symmetry and Transformations

11.8.10 Recognise and use symmetry, particularly odd and even functions, to simplify and solve integration problems

When integrating, it's important to also look at symmetry, as symmetry properties can make solving questions easier and faster.

Even Functions	Odd Functions	Periodic Functions (period = a)

THEORY

One particular thing to note about odd functions is that, if we wish to find the *area under the curve* between $-a \leq x \leq a$, we can simply evaluate the integral $\left|\int_0^a f(x)dx\right|$ and then multiply that by 2.

EXAMPLE QUESTION

Solve the following integrals.

a. $\int_{-\pi}^{\pi} \sin^3 x + \sin x \, dx$

(1 mark)

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b. $\int_0^{2\pi} \cos^2 x \, dx$

(2 marks)

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c. $\int_{-1}^{1} (x+1)(1-x)(x+3)(3-x) \, dx$

(1 mark)

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Some questions may give you the value of an integral $\int_a^b f(x)\,dx$ and ask you to consider how a shift or dilation in a function affects the value. There are 4 main transformations that questions may ask:

1. Vertical Translation - $\int_a^b f(x) + k \, dx$

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2. Horizontal Translation - $\int_{a-k}^{b-k} f(x-k) \, dx$

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THEORY

3. Vertical Dilation - $\int_a^b k\,f(x)\,dx$

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4. Horizontal Dilation - $\int_{\frac{a}{k}}^{\frac{b}{k}} f\,(kx)\,dx$

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EXAMPLE QUESTION

Given that $\int_{-1}^{2} \boldsymbol{x\,(x+1)(x-2)\,dx} = -\frac{9}{4}$, find the value of the following integrals:

a. $\int_{-1}^{2} x\,(x+1)(x-2)\ +3\,dx$

STEP 1: Determine the shift

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STEP 2: Evaluate the Integral

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b. $\int_0^3 x\,(x-1)(x-3)\,dx$

STEP 1: Determine the shift

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STEP 2: Evaluate the Integral

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c. $\int_{-1}^{2} \frac{1}{3} x(x+1)(x-2)\,dx$

STEP 1: Determine the shift

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STEP 2: Evaluate the Integral

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d. $\int_{-\frac{1}{4}}^{\frac{1}{2}} 4\,x(4x+1)(4x-2)\,dx$

THEORY

STEP 1: Determine the shift

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STEP 2: Evaluate the Integral

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11.8.12 Use an online computational application to evaluate definite and indefinite integrals involving functions within and beyond the scope of the Mathematics Advanced course

Some online tools can aid us in solving integrals, whether they be within or beyond the Mathematica Advanced syllabus. These include:

Integral Calculator	Symbolab	Mathway	Wolfram Alpha

Please see homework problems to practice the content in this dotpoint!

Integration in the Real World

11.8.13 Model and solve practical problems involving integrals and areas of regions bounded by a curve and the x-axis, or by a curve and the y-axis, involving functions within the scope of the Mathematics Advanced course

Much like rates of change, integration can also be used to solve practical problems involving area and accumulation, such as:

- Total distance travelled
- Volume or flow of water
- Total growth/decay over time

We can continue to use properties of definite integrals and concepts such as symmetry to help us solve these problems.

THEORY

EXAMPLE QUESTION

A tap is leaking water, so Bob decides to devote his life to tightening the tap in an attempt to stop the tap from leaking. Unfortunately, the tap leaks at a rate of $\frac{dV}{dt} = 10e^{-\frac{t}{3}}\, ml\, min^{-1}$ at t minutes.

a. How much water has leaked after 5 minutes?

STEP 1: Integrate $\frac{dV}{dt}$ to find V

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STEP 2: Sub in initial condition to find C

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STEP 3: Sub in t = 5

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b. What's the maximum amount of water that will theoretically leak?

STEP 1: Sub ∞ into V

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PRACTICE QUESTION

1. **During an engine cycle, the rate of fuel flow (mL/s) into a tank is given by** $t(3-t)^2, \quad 0 \le t \le 3$

Find the total volume of fuel injected during one tank.

(1 mark)

THEORY

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Wrap up

Today we learnt about the various applications of integration in concepts such as finding areas and in solving real-life problems.

We also explored how concepts such as symmetry and transformation may affect our integrals, as well as how Properties of Definite Integrals derived from the FTC can simplify solving integrals.

Next week, we'll be looking at continuous random variables and how to find their variance and expected values.

FORMULA SHEET

Fundamental Theorem of Calculus

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Area under a graph

- Graph and the x-axis (above x-axis):

$$A = \int_a^b f(x)\,dx$$

- Graph and the x-axis (below x-axis):

$$A = \left| \int_a^b f(x)\,dx \right| \text{ or } A = -\int_a^b f(x)\,dx$$

- Graph and the y-axis:

$$A = \int_c^d f(y)\,dy$$

Symmetry and Transformations

- Even functions:

$$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$$

- Odd functions:

$$\int_{-a}^a f(x)\,dx = 0$$

- Periodic (period = a):

$$\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

- Vertical Translation:

$$\int_a^b (f(x) + k)\,dx = \int_a^b f(x)\,dx + k(b - a)$$

FORMULA SHEET

- Horizontal Translation:

$$\int_{a-k}^{b-k} f(x-k)\,dx = \int_a^b f(x)\,dx$$

- Vertical Dilation:

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$

- Horizontal Dilation ($k \neq 0$):

$$\int_{a/k}^{b/k} f(kx)\,dx = \frac{1}{k}\int_a^b f(x)\,dx$$

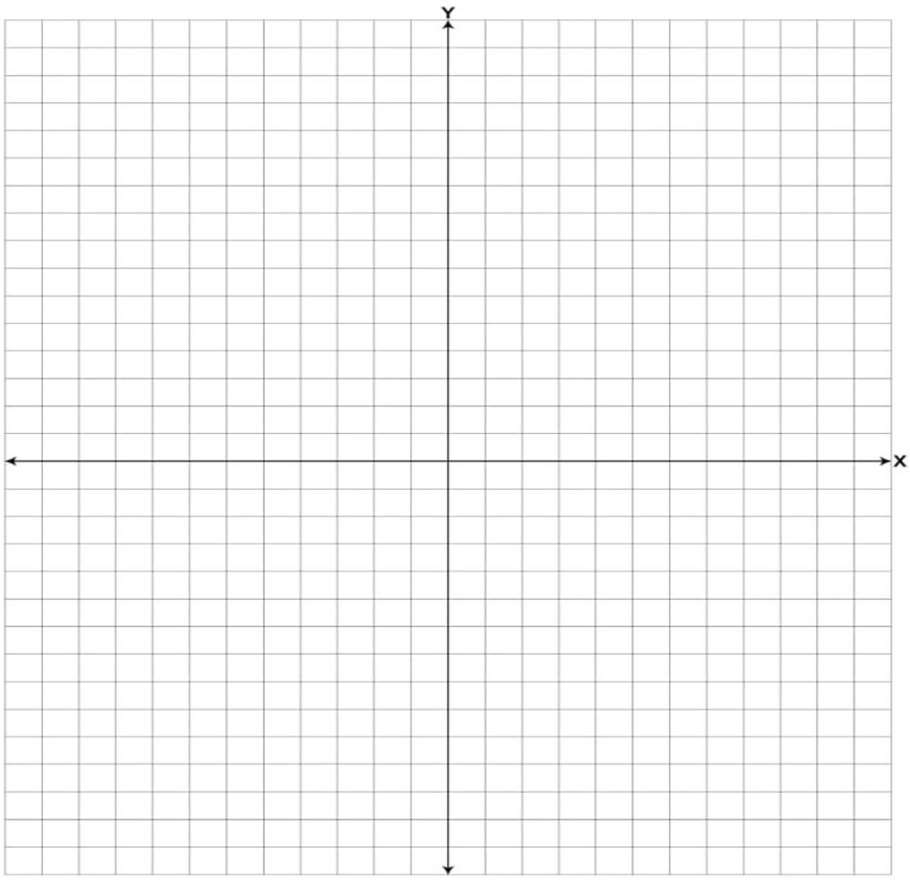
EXTRA QUESTIONS

1. The rate of fuel consumption for a car is given by the function $f(x) = \frac{2}{x+1}$ L/km. How much fuel is used between 1km and 4km?

   (2 marks)

2. Calculate the area under the curve $y = 4sinx$ between $x = 5$ and $x = 2\pi$.

   (3 marks)



EXTRA QUESTIONS

3. Consider the functions $f(x) = sin^2x$ and $g(x) = cos^2x$.
   What is the value of $\int_0^{\frac{\pi}{2}} f(x)\,dx + \int_0^{\frac{\pi}{2}} g(x)\,dx$?

   (4 marks)

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4. Find the value of $\int_0^2 cos2x\,dx - 2\int_0^2 cos^2x\,dx$.

   (2 marks)

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5. Considering the piecewise function, find the value of $I = \int_4^0 f(x)\,dx$.

$$f(x) = \begin{cases} x^2 - 4x & 0 \leq x \leq 2 \\ 6 - x & 2 < x \leq 4 \end{cases}$$

   (2 marks)

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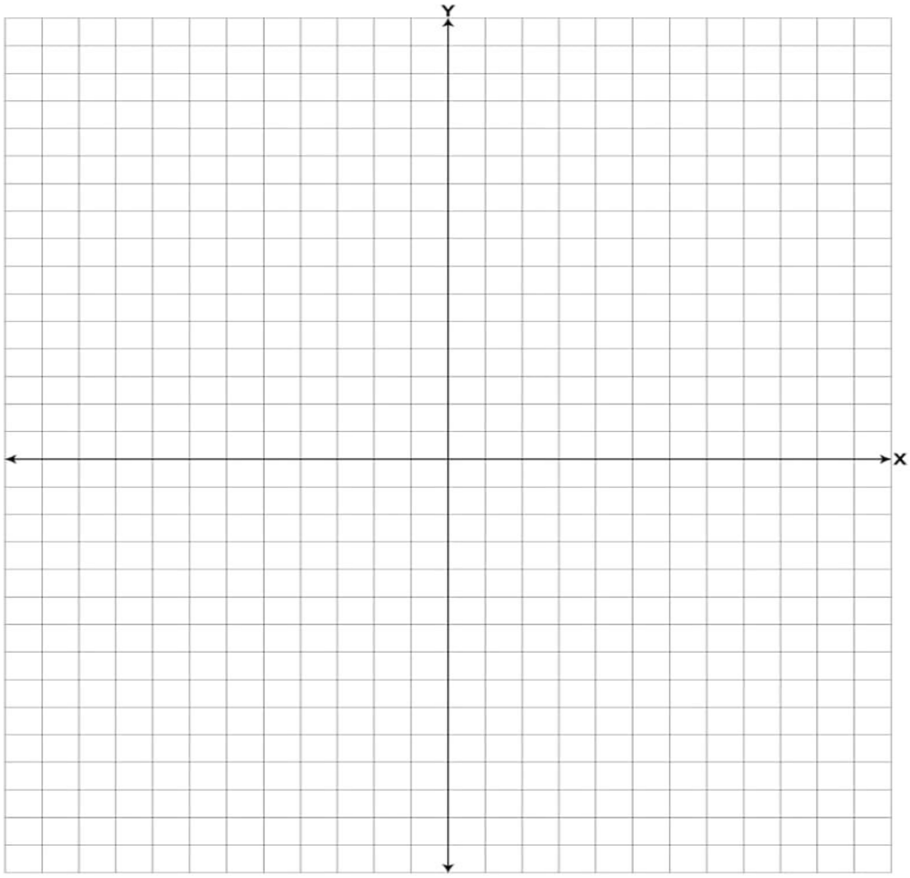
EXTRA QUESTIONS

6. Find the area between the parabolas $y = (x-3)(x-5)$ and $y = (2-x)(x-6)$

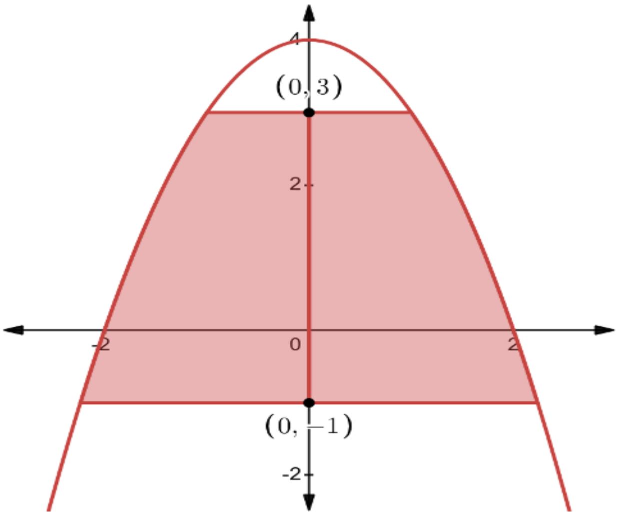
(2 marks)



7. Given the graph of a parabola, find the area of the shaded region.

(2 marks)

EXTRA QUESTIONS



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8. Given that $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} f(x)\,dx = 7$, where f(x) is a continuous, even and periodic function with a period of $\frac{\pi}{2}$, find the value of:

(1 mark each)

a. $\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} f(x)\,dx$

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b. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2f(x) + 3\,dx$

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c. $\int_{-\frac{3\pi}{4}}^{-\frac{\pi}{4}} f(x)\,dx + \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} f(x)\,dx$

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d. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} f(2x)\, dx$

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9. The tension T(x) N in a hanging cable is modelled by

$$T(x) = 5 + \frac{x^2}{4}$$

The total tension can be calculated by the area under curve. Find the total tension across the interval $-3 \leq x \leq 3$.

(1 mark)

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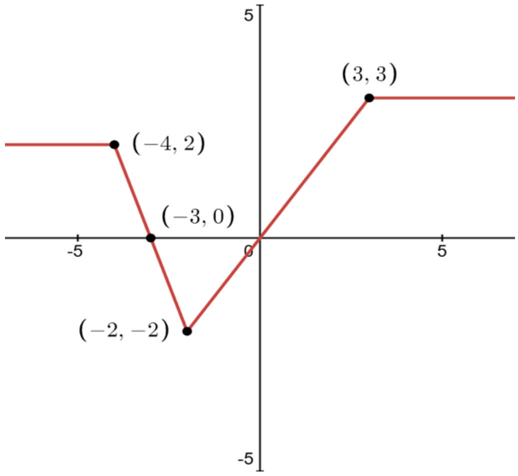
HOMEWORK

1. The graph of f(x) is given below. What is the value of $\int_{-4}^{4} \mathrm{f(x)}\ \mathrm{dx}$?



a. $\frac{11}{2}$
b. $\frac{25}{2}$
c. $\frac{13}{2}$
d. $\frac{5}{2}$

(1 mark)

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2. If $\int_0^2 f(x)\ dx = 5$ and $\int_0^2 g(x)\ dx = 3$, find the value of $\int_0^2 2f(x) - 3g(x)\, dx$.

a. 1
b. -1
c. 2
d. -2

(1 mark)

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3. Which of the following is equivalent to $\int (sinx + sin3x)cosx\, dx$?

a. $sin4x + C$
b. $\frac{sin2x}{2} + \frac{sin4x}{8} + C$
c. $-\frac{cos2x}{2} - \frac{cos4x}{8} + C$
d. $cos4x + C$

(1 mark)

HOMEWORK

4. What is $\frac{d}{dx}[\int_a^b f(x)\ dx]$ equal to?

a. 0
b. $f(x)$
c. $f(a)$
d. $f(b)$

(1 mark)

5. Consider the trigonometric functions $sinx$ and $cosx$ and the two points of intersection with the smallest positive x-coordinates. Which of the following is equivalent to finding the area between these two functions within the points of intersection?

a. $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} sinx + cosx\ dx$
b. $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sqrt{2}sin(x - \frac{\pi}{4})\ dx$
c. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{2}cosx\ dx$
d. $\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} cosx - sinx\ dx$

(1 mark)

HOMEWORK

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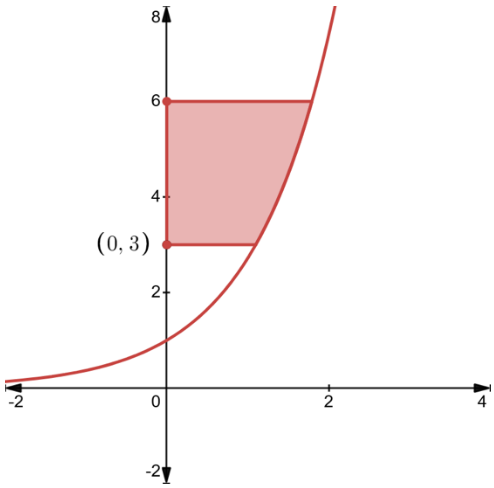
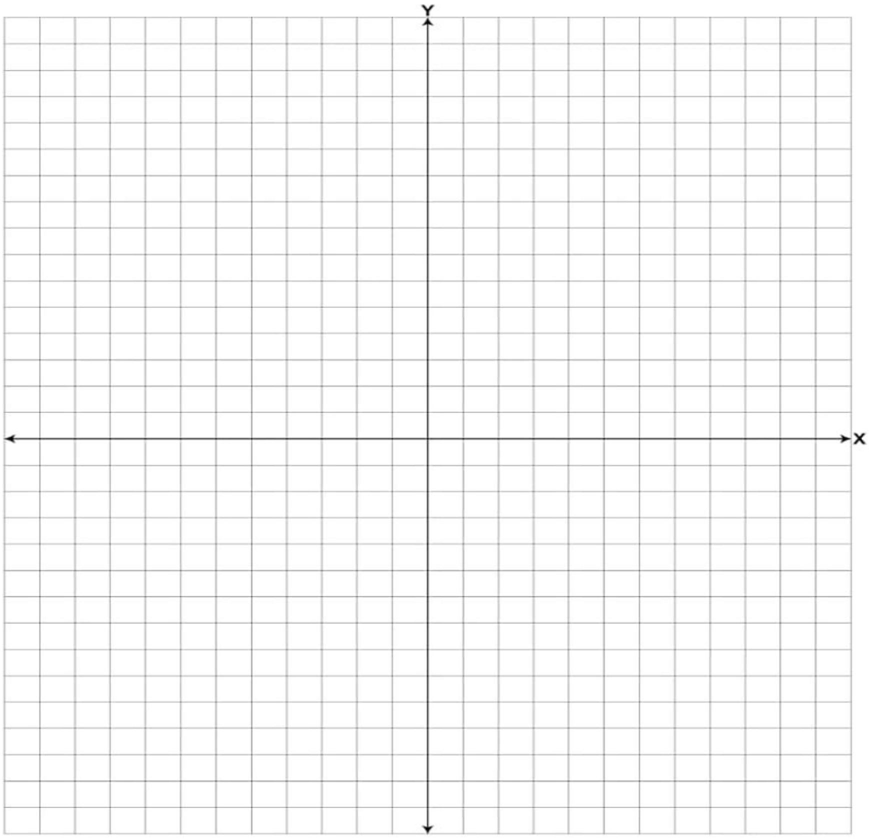
6. **Find the area of the shaded regions for each graph, given its equation.**



a. $y = e^x$

(3 marks)

HOMEWORK



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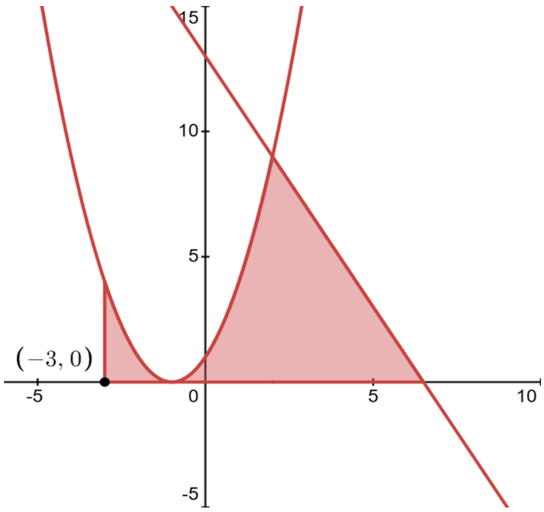
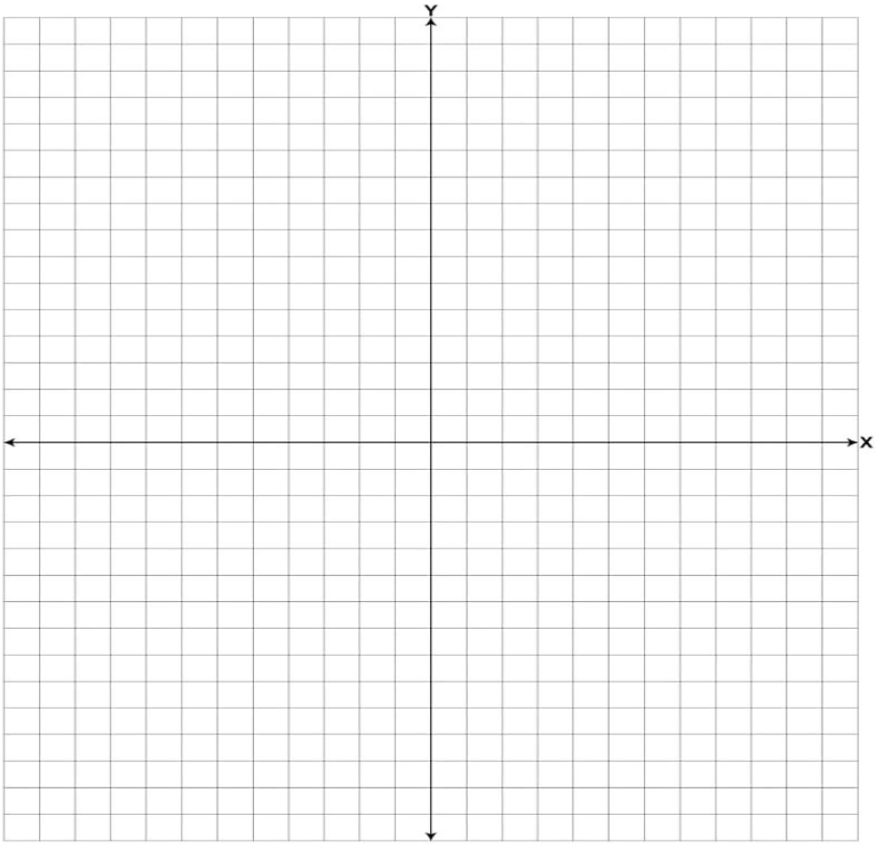


b. $y = (x + 1)^2$ and $y = -2x + 13$

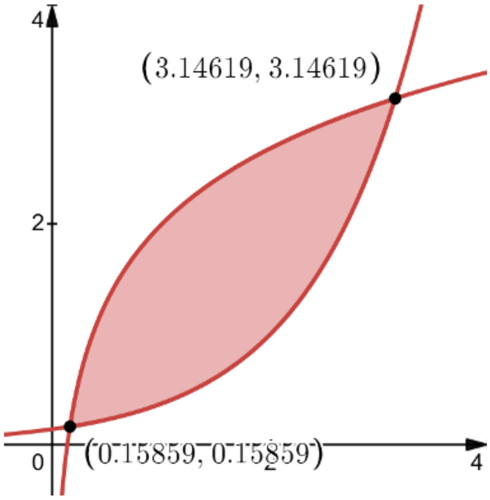
(3 marks)

HOMEWORK



HOMEWORK

7. Considering the relationship between $y = e^{x-2}$ and $y = lnx + 2$, find the area of the shaded region.



(3 marks)

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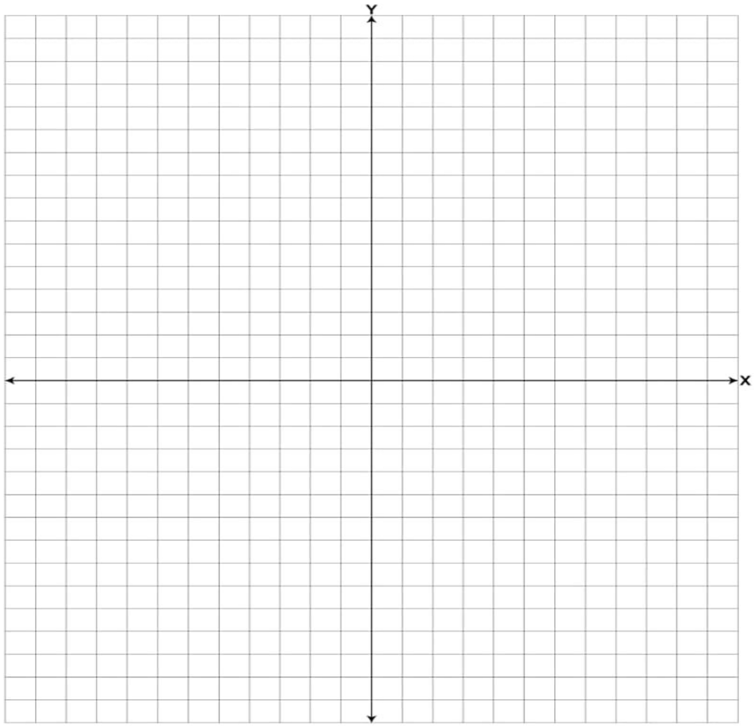
8. The rate of change of a plant's height is given by

$$\frac{dh}{dt} = 5ln(t+1)\ cm\ day^{-1}$$

By drawing a graph, find the change in height after the first ten days.

(3 marks)

HOMEWORK



9. Given $\int_0^{\frac{\pi}{4}} dx = \frac{\pi}{4}$ and $\int_0^{\frac{\pi}{2}} sinx \ dx = 1$, without directly integrating solve the following:

 a. $\int_0^{\frac{\pi}{4}} (sinx - cosx)^2 \ dx$

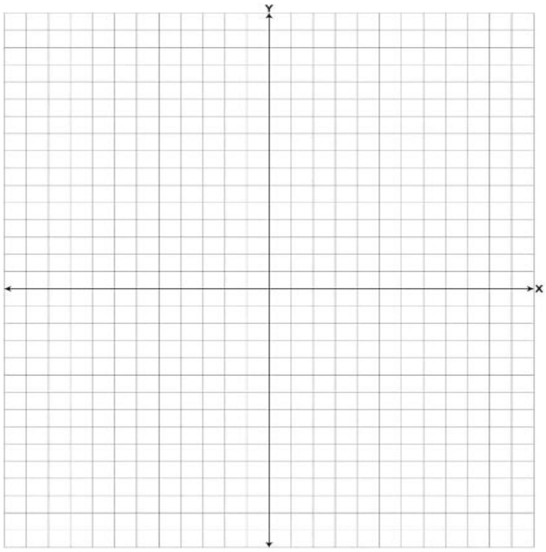
(2 marks)

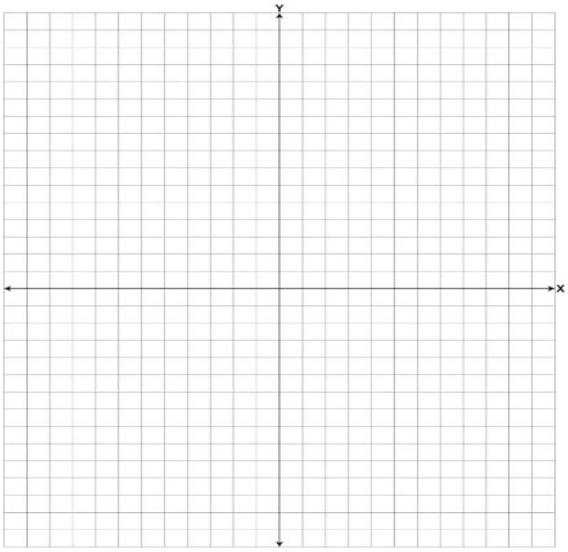
HOMEWORK

b. $\int_0^{\frac{\pi}{4}} sin^2x \ dx$

(2 marks)

(Hint: consider its graph and the graph of $\int_0^{\frac{\pi}{4}} \quad cos^2x \ dx$)



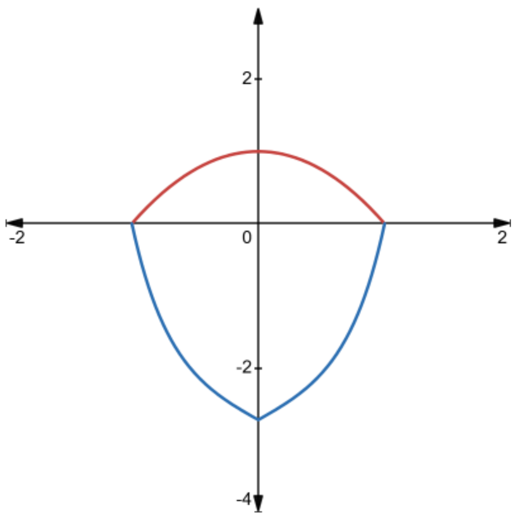


10. Consider the function $f(x) = 3 + cosx$. Show that $\int_{-\pi}^{\pi} f\,(x)\ dx = 2\int_0^{\pi} f\,(x)\ dx$, and explain why this symmetry property holds.

(3 marks)

HOMEWORK

11. **Consider the functions $f(x) = 1 - x^2$ and $g(x) = |xe^{x^2}| - e$.**



a. Identify the graph above the x-axis.

(1 mark)

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b. Find the points of intersections.

(2 marks)

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c. Hence, find the area between the two graphs.

(3 marks)

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HOMEWORK

CHALLENGE QUESTION

1. Find $\frac{d}{dx}(xtan^{-1}(x) - \frac{ln(x^2+1)}{2})$

(1 mark)

2. Hence, find the area under the graph of $y = tan^{-1}(x/2)$ from $x = \pi$ to $x = 2\pi$

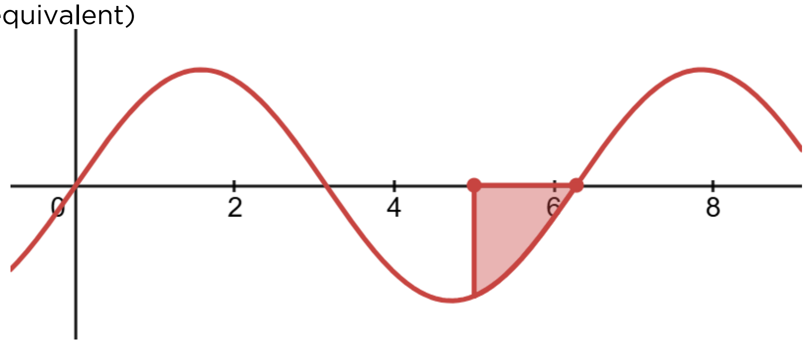
(3 marks)

EQ ANSWERS

1.

1	Integrates f(x) $f(x) = \frac{2}{x+1}$ $\int_1^4 f(x)dx = \int_1^4 \frac{2}{x+1}\,dx$ $F(x)\|_1^4 = 2\ln\|x+1\|\Big\|_1^4$
1	Substitutes bounds $F(x)\|_1^4 = 2\ln\|x+1\|\Big\|_1^4$ $= 2\ln(5) - 2\ln 2$ $= 2\ln\frac{5}{2}$ $= 1.83\ \text{L}$

2.

1	Draws graph to determine whether intervals are below or above axis (or equivalent) 
1	Integrates equation (with absolute limit) $A = \left\|\int_a^b f(x)\,dx\right\|$ $= \left\|\int_5^{2\pi} 4\sin x\ dx\right\|$ $= 4\|[-\cos x]\|_5^{2\pi}$
1	Substitutes bounds to find area $= 4\|[-\cos x]\|_5^{2\pi}$ $= 4\|-1+0.281\|$ $= 2.87\ u^2$

3.

1	Sub function and bounds into $\int_a^b \boldsymbol{f(x)\,dx}$ $\int_0^{\frac{\pi}{2}} \sin^2 x\,dx + \int_0^{\frac{\pi}{2}} \cos^2 x\,dx$
1	Add integrals together

	Bounds are the same so we can add the integrals together $$= \int_0^{\frac{\pi}{2}} \sin^2 x\,dx + \int_0^{\frac{\pi}{2}} \cos^2 x\,dx$$ $$= \int_0^{\frac{\pi}{2}} \sin^2 x\,dx + \cos^2 x\,dx$$ $$= \int_0^{\frac{\pi}{2}} dx$$	
1	Integrate function $$= \int_0^{\frac{\pi}{2}} dx$$	$$= x\Big\|_0^{\frac{\pi}{2}}$$
1	Substitute bounds $$= x\Big\|_0^{\frac{\pi}{2}}$$ $$= \frac{\pi}{2}$$	

4.

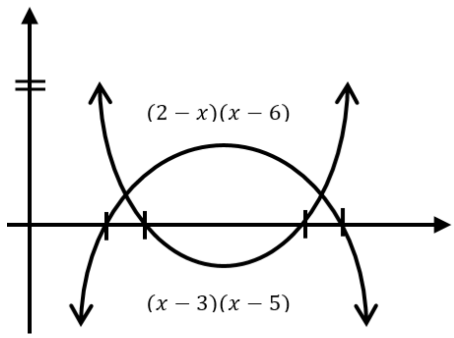
1	Adds integrals together and integrates $$\alpha = \int_0^2 \cos 2x\,dx - 2\int_0^2 \cos^2 x\,\,dx$$ $$= \int_0^2 \cos 2x - 2\cos^2 x\,\,dx$$ $$\cos 2x = 2\cos^2 x - 1$$ $$1 + \cos 2x = 2\cos^2 x$$ $$\therefore \alpha = \int_0^2 \cos 2x - \cos 2x - 1\,dx$$ $$= -\int_0^2 dx$$ $$= -x\|_0^2$$
2	Substitute bounds $$= -x\|_0^2$$ $$\therefore \alpha = -2$$

5.

1	Integrates separately $$I = \int_4^0 f(x)dx$$ $$I = -\int_0^4 f(x)dx$$ $$-I = \int_0^4 f(x)dx$$ $$= \int_0^2 f(x)dx + \int_2^4 f(x)dx$$ $$= \int_0^2 x^2 - 4x\,dx + \int_2^4 6 - x\,dx$$

	$= \left[\frac{x^3}{3} - 2x^2\right]_0^2 + \left[6x - \frac{x^2}{3}\right]_2^4$
1	Substitute bounds $= \left[\frac{x^3}{3} - 2x^2\right]_0^2 + \left[6x - \frac{x^2}{3}\right]_2^4$ $= \left(\frac{8}{3} - 8\right) - (0) + \left(24 - \frac{16}{2}\right) - \left(12 - \frac{4}{2}\right)$ $= \frac{2}{3}$ $\therefore I = -\frac{2}{3}$

6.

1	Sketches both graphs and finds Points of Intersections  finding POI: $(2-x)(x-6) = (x-3)(x-5)$ $= -x^2 + 8x - 12 = x^2 - 8x + 15$ $2x^2 - 16x + 27 = 0$ $\Delta = (-16)^2 - 4(2)(27)$ $= 40$ $x = 16 \pm \frac{\sqrt{40}}{4}$ $= 4 \pm \frac{\sqrt{10}}{2}$
1	Integrates to find area, using POI as bounds Let $b = 4 + \frac{\sqrt{10}}{2}, a = 4 - \frac{\sqrt{10}}{2}$ $A = \int_a^b f(x) - g(x)\, dx$ $= \int_a^b -2x^2 + 16x - 27\, dx$ $= \left[-\frac{2x^3}{3} + 8x^2 - 27x\right]_a^b$ $= (-17.40) - (-27.94)$ $\therefore A = 10.54\ u^2$

7.

1	Finds equation (and rearranges in terms of y) Parabola: Vertex at (0,4), x-intercepts at -2 and 2 $y = a(x-b)^2 + c$ $y = a(x)^2 + 4$ $y = -x^2 + 4$ $x^2 = 4 - y$ $x = \pm\sqrt{4-y}$
1	Integrates with respect to y to find area Consider $x = \sqrt{4-y}$ $A_1 = \int_{-1}^{3} x\ dy$

EQ ANSWERS

	$$A_1 = \int_{-1}^{3} \sqrt{4-y}\ dy$$ $$A_1 = [\frac{(4-y)^{\frac{3}{2}}(-1)}{\frac{3}{2}}]^3{}_{-1}$$ $$A_1 = [\frac{-2(4-y)^{\frac{3}{2}}}{3}]^3{}_{-1}$$ $$A_1 = -\frac{2}{3}[(4-3)^{\frac{3}{2}} - (4+1)^{\frac{3}{2}}]$$ $$A_1 = 6.79$$ Therefore, total area is $2 \times A_1 = 13.57$, as we must account for the area to the left of the y-axis

8a.

1	Finds correct value The interval $-\frac{\pi}{4}$ to $\frac{3\pi}{4}$ is two consecutive periods $\rightarrow$ $2 \times 7 = 14$

8b.

1	Finds correct value $$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2\,\mathrm{f}(\mathrm{x}) + 3\ \mathrm{dx} = 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \mathrm{f}(\mathrm{x})\ \mathrm{dx} + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 3\ \mathrm{dx}$$ $$= 14 + 3(\frac{\pi}{4} - (-\frac{\pi}{4}))$$ $$= 14 + 3\frac{\pi}{2}$$

8c.

1	Finds correct value Both intervals are of length $\frac{\pi}{2} \rightarrow 7 + 7 = 14$

8d..

1	Finds correct value $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \mathrm{f}(2\mathrm{x})\ \mathrm{dx} \rightarrow$ horizontal dilation $\rightarrow \int_{\frac{a}{k}}^{\frac{b}{k}} \mathrm{f}(\mathrm{kx})\ \mathrm{dx} = \frac{1}{k}\int_a^b \mathrm{f}(\mathrm{x})\ \mathrm{dx}$ $$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \mathrm{f}(2\mathrm{x})\ \mathrm{dx} = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{f}(\mathrm{x})\ \mathrm{dx}$$ Interval of $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ has two consecutives periods $\rightarrow 2 \times 7$ Therefore $\frac{1}{2} \times 2 \times 7 = 7$

9.

1	Integrates to find tension $$\int_{-3}^{3} \mathrm{I}(\mathrm{x})\mathrm{dx} = \int_{-3}^{3} 5 + \frac{\mathrm{x}^2}{4}\mathrm{dx}$$ $$= 2\int_0^3 5 + \frac{\mathrm{x}^2}{4}\mathrm{dx}$$ $$\left(\text{as } 5 + \frac{\mathrm{x}^2}{4} \text{ is even}\right)$$ $$= 2\left[5\mathrm{x} + \frac{\mathrm{x}^3}{12}\right]_0^3$$ $$= 2\left[\left(15 + \frac{27}{12}\right) - (0)\right]$$ $$= \frac{69}{2}\mathrm{N}$$

HW ANSWERS

Module 6 | Lesson 7: Further Integration II

1.

1	c. $-\ln 2$

2.

1	b. $-\cos(e^x) + C$

3.

1	a. $-\cos(\ln x) + C$

4.

1	c. $u = \ln(e^{2x} + 1)$

5.

1	a. 0

6.

1	$u = \cos 4x$ $du = -4\sin 4x\,dx$	Alternatively $\int \frac{\sin(4x)}{\cos^3(4x)}dx$ $\int \tan(4x).\sec^2(4x)\,dx$ $= \frac{1}{8}\tan^2 x$ Where $u = \tan(4x)$
1	$\int \frac{\sin(4x)}{\cos^3(4x)}dx = \int \frac{-\frac{1}{4}}{u^3}du$ $= -\frac{1}{4}\int u^{-3}\,du$	
1	$-\frac{-1}{4} \times \frac{-1}{2}u^{-2} + C$ $= \frac{1}{8}u^{-2} + C$	
1	$\frac{1}{8\cos^2(4x)} + C$	

7.

1	$u = \cos 4x$ $du = -4\sin 4x\,dx$	Alternatively $\int \frac{\sin(4x)}{\cos^3(4x)}dx$ $\int \tan(4x).\sec^2(4x)\,dx$ $= \frac{1}{8}\tan^2 x$ Where $u = \tan(4x)$
1	$\int \frac{\sin(4x)}{\cos^3(4x)}dx = \int \frac{-\frac{1}{4}}{u^3}du$ $= -\frac{1}{4}\int u^{-3}\,du$	
1	$-\frac{-1}{4} \times \frac{-1}{2}u^{-2} + C$ $= \frac{1}{8}u^{-2} + C$	

1	$\frac{1}{8\cos^2(4x)} + C$	

8a.

1	$u = \sin(6x+1)$ $du = 6\cos(6x+1)\,dx$ $\cos(6x+1)dx = \frac{1}{6}du$
1	$\int \cos(6x+1)\sin(6x+1)dx$ $= \int u \cdot \frac{1}{6}du$ $= \frac{1}{12}u^2 + C$
1	$= \frac{1}{12}\sin^2(6x+1) + C$

8b.

1	Yes, both can be valid if the expressions only differ by a constant
1	$\frac{1}{12}\sin^2(6x+1) - \left(-\frac{1}{12}\cos^2(6x+1) + C\right)$ $= \frac{1}{12}(\sin^2(6x+1) + \cos^2(6x+1))$ $= \frac{1}{12}(1) = \frac{1}{12}$
1	Hence both answer differ by a constant value of $\frac{1}{12}$, so they are both valid solution to the integral.

9.

1	$u = \ln(x^2+1)$ $du = \frac{2x}{x^2+1}dx$
1	$\int \frac{2x}{x^2+1}\cos(\ln(x^2+1))dx$ $= \int \cos u \; du$ $= \sin u + C$
1	$= \sin(\ln(x^2+1)) + C$

10a..

1	$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$
1	$= -\ln\lvert\cos x\rvert + C$ $\therefore$ $-\ln\lvert\cos x\rvert$ is primitive of $\tan x$

10b.

1	$u = \frac{1}{x}$ $du = -\frac{1}{x^2}dx$

HW ANSWERS

	$\frac{1}{x^2}dx = -du$
1	$= \ln\left\|\cos\left(\frac{1}{x}\right)\right\| + C$

CQ1.

1	$\int \frac{1}{e^x+1} = \int \frac{e^{-x}}{1+e^{-x}}dx$
1	$u = 1 + e^{-x}$ $du = -e^{-x}dx$ $e^{-x}dx = -du$
1	$\int \frac{1}{e^x+1}dx = \int -\frac{1}{u}du$ $= -\ln\|u\| + C$
1	$= -\ln\|1+e^{-x}\| + C$

CQ2a.

1	$\frac{-\frac{-\sin x}{\cos^2 x} + \sec^2 x}{\sec x + \tan x}$ $= \frac{\tan x \sec x + \sec^2 x}{\sec x + \tan x}$ $= \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}$ $= \sec x$

CQ2b.

1	$= \int \sec^3 x\,(\sec x)\ dx$ $= \int (\sec x)(\tan^2 x + 1)\ dx$ $= \sec\tan^2 x + \sec x\ dx$
1	$= \int \frac{\sin x}{\cos^3 x} + \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}$ $= \frac{1}{2}(\cos x)^{-2} + \ln\|\sec x + \tan x\| + C$

elevate